Mathematics 325 Exam III Name: Dr. Grow

Summer 2012

1.(30 pts.) Solve the eigenvalue problem $X''(x)+\lambda X(x)\overset{(1)}{=}0,\quad X'(0)\overset{(2)}{=}0\overset{(3)}{=}X'(1)$. You may assume that λ is a real number, but for full credit you must show all details for the cases $\lambda>0$, $\lambda=0$, and $\lambda<0$.

(15 pts.) Case $\lambda>0$ (say $\lambda=\beta^2$ where $\beta>0$): Then (1) becomes $X''(x)+\beta^2X(x)=0$. If $X(x)=e^{rx}$ for some constant r then $X'(x)=re^{rx}$ and $X''(x)=r^2e^{rx}$ so substituting into (1) gives $r^2e^{rx}+\beta^2e^{rx}=0$. Dividing through by e^{rx} leads to $r^2+\beta^2=0$ and hence $r=\pm\sqrt{-\beta^2}=\pm i\beta$. The general solution of (1) is $X(x)=\tilde{c}_1e^{i\beta x}+\tilde{c}_2e^{-i\beta x}$ where $\tilde{c}_1$ and $\tilde{c}_2$ are constants. Using the Euler identity $e^{i\theta}=\cos(\theta)+i\sin(\theta)$, we can express the solution of (1) as $X(x)=\tilde{c}_1(\cos(\beta x)+i\sin(\beta x))+\tilde{c}_2(\cos(\beta x)-i\sin(\beta x))=(\tilde{c}_1+\tilde{c}_2)\cos(\beta x)+i(\tilde{c}_1-\tilde{c}_2)\sin(\beta x)$

(4) $=c_1\cos(\beta x)+c_2\sin(\beta x)$ where c_1 and c_2 are arbitrary constants. Note then that $X'(x)$

(2) $=-\beta c_1\sin(\beta x)+\beta c_2\cos(\beta x)$ so (2) implies $0=X'(0)=\beta c_2$ so $c_2\overset{(1)}{=}0$. Also (3) implies $0=X'(1)=-\beta c_1\sin(\beta)+\beta \overset{0}{c_2}\cos(\beta)=-\beta c_1\sin(\beta)$. (2) In order for a nonzero solution $X=X(x)$ to exist, we must have $c_1\neq0$ and hence $\sin(\beta)=0$. (1) Thus $\beta=\beta_n=n\pi$ (2) for $n=1,2,3,\ldots$. In this case, the corresponding solution of (1)-(2)-(3) with $\lambda_n=\beta_n^2=(n\pi)^2$ (1) is $X_n(x)=\cos(n\pi x)$, (2) up to a constant factor.

(5 pts.) Case $\lambda=0$: Then (1) becomes $X''(x)=0$ so two integrations give $X(x)\overset{(2)}{=}c_1x+c_2$ where c_1 and c_2 are arbitrary constants. Since $X'(x)=c_1$ both (2) and (3) are satisfied if $c_1\overset{(1)}{=}0$. Since c_2 is arbitrary we have the nontrivial solution $X_0(x)=1$ (2) (up to a constant factor) of (1)-(2)-(3) when $\lambda_0=0$.

(10 pts.) Case $\lambda<0$ (say $\lambda=-\beta^2$ where $\beta>0$): Then (1) becomes $X''(x)-\beta^2X(x)=0$. If $X(x)=e^{rx}$ for some constant r then, as in the case $\lambda>0$, substituting into (1) yields $r^2e^{rx}-\beta^2e^{rx}=0$ and hence $r^2-\beta^2=0$ or $r=\pm\beta$. The general solution of (1) is $X(x)=\tilde{c}_1e^{\beta x}+\tilde{c}_2e^{-\beta x}$ where $\tilde{c}_1$ and $\tilde{c}_2$ are arbitrary constants. It is convenient to express this solution in terms of the hyperbolic functions: $\cosh(t)=\frac{e^t+e^{-t}}{2}$ and $\sinh(t)=\frac{e^t-e^{-t}}{2}$. Note that $\cosh(t)+\sinh(t)=e^t$ and $\cosh(t)-\sinh(t)=e^{-t}$ so

$$X(x)=\tilde{c}_1(\cosh(\beta x)+\sinh(\beta x))+\tilde{c}_2(\cosh(\beta x)-\sinh(\beta x))=(\tilde{c}_1+\tilde{c}_2)\cosh(\beta x)+(\tilde{c}_1-\tilde{c}_2)\sinh(\beta x)$$

(4) and thus $X(x)=c_1\cosh(\beta x)+c_2\sinh(\beta x)$ for arbitrary constants c_1 and c_2.

(OVER)

Case $\lambda<0$ (cont.): Differentiating we have $X'(x) \overset{(2)}{=} \beta c_1 \sinh(\beta x) + \beta c_2 \cosh(\beta x)$.

Applying ② yields $0 = X'(0) = \beta c_2$ so $c_2 \overset{(1)}{=} 0$. Applying ③ gives

$0 = X'(1) = \beta c_1 \sinh(\beta) + \beta \cancel{c_2}^{\,0} \cosh(\beta) = \beta c_1 \sinh(\beta)$. But $\sinh(\beta) > 0$ for $\beta > 0$

so we must have $c_1 \overset{(2)}{=} 0$. Therefore $X(x) = c_1 \cosh(\beta x) + c_2 \sinh(\beta x) \equiv 0$. There are no nontrivial (i.e. not identically zero) solutions to ①-②-③ when $\lambda < 0$. (1)

Summarizing, the eigenvalues and corresponding eigenfunctions of ①-②-③ are

$$\boxed{\begin{array}{l} \lambda_0 = 0, \quad X_0(x) = 1 \\ \lambda_n = (n\pi)^2, \quad X_n(x) = \cos(n\pi x) \qquad (n = 1, 2, 3, \ldots). \end{array}}$$

2.(35 pts.) Solve $u_{xx}+u_{yy}\stackrel{(1)}{=}0$ if $0<x<1,\ 0<y<\infty$, subject to the homogeneous boundary conditions $u_x(0,y)\stackrel{(2)}{=}0\stackrel{(3)}{=}u_x(1,y)$ if $y\geq 0$ and $u_y(x,0)\stackrel{(4)}{=}0$ if $0\leq x\leq 1$, and the nonhomogeneous boundary condition $u(x,0)\stackrel{(5)}{=}\cos^2(\pi x)$ if $0\leq x\leq 1$.

Suggestions: The eigenvalue problem in # 1 and the identity $\cos^2(\theta)=(1+\cos(2\theta))/2$ may be useful.

Bonus (10 pts.): Is there more than one classical solution to # 2? Give reasons for your answer.

We use the method of separation of variables. We seek nontrivial solutions of the homogeneous portion of the problem (1)-(2)-(3)-(4) of the form $u(x,y)=X(x)Y(y)$. (1)
Substituting this functional form for u into (1) gives $X''(x)Y(y)+X(x)Y''(y)=0$. (1)
Rearranging yields
$$\frac{Y''(y)}{Y(y)}=-\frac{X''(x)}{X(x)}=\text{constant}=\lambda. \quad (4)$$

Substituting the assumed functional form for u into (2) gives
$$0=u_x(0,y)=X'(0)Y(y) \quad \text{for all } y\geq 0. \quad (1)$$

If $X'(0)\neq 0$ then the above equation implies $Y(y)=0$ for all $y\geq 0$ and hence $u(x,y)=X(x)Y(y)=0$ for all $0\leq x\leq 1$ and $0\leq y<\infty$, contradicting nontriviality of u. Therefore $X'(0)\stackrel{(1)}{=}0$ if we are to have nontrivial solutions of (1)-(2)-(3)-(4). In a similar manner we obtain from (3) and (4) that $X'(1)\stackrel{(1)}{=}0$ and $Y'(0)\stackrel{(1)}{=}0$ if we are to have nontrivial solutions u of (1)-(2)-(3)-(4). Summarizing, we must have

$$\begin{cases} (2)\quad \boxed{X''(x)+\lambda X(x)\stackrel{(6)}{=}0,\quad X'(0)\stackrel{(7)}{=}0\stackrel{(8)}{=}X'(1),} \\ (2)\quad Y''(y)-\lambda Y(y)\stackrel{(9)}{=}0,\quad Y'(0)\stackrel{(10)}{=}0, \end{cases}$$

4 pts. to here

where λ is a constant. The boxed eigenvalue problem in the system above was solved in problem 1. There we obtained eigenvalues $\lambda_n=(n\pi)^2$ $(n=0,1,2,\ldots)$ and corresponding eigenfunctions $X_n(x)=\cos(n\pi x)$ $(n=0,1,2,\ldots)$. Substituting $\lambda=\lambda_n=(n\pi)^2$ into (9)-(10) yields

$$(2)\quad Y_n''(y)-(n\pi)^2Y_n(y)\stackrel{(9')}{=}0,\quad Y_n'(0)\stackrel{(10')}{=}0.$$

By work in cases $\lambda=0$ and $\lambda<0$ in problem 1, the general solution of (9') is

$$(1)\quad Y_0(y)=c_1y+c_2 \quad \text{if } n=0,$$

$$(1)\quad Y_n(y)=c_1\sinh(n\pi y)+c_2\cosh(n\pi y) \quad \text{if } n\geq 1.$$

Applying (10') gives

(OVER)

#2 (cont.):

$$0 = Y_0'(0) = c_1 \qquad \text{if } n=0,$$

$$0 = Y_n'(0) = n\pi c_1 \overbrace{\cosh(0)}^{1} + n\pi c_2 \overbrace{\sinh(0)}^{0} = n\pi c_1 \qquad \text{if } n \geq 1.$$

Therefore $\overset{(1)}{c_1} = 0$ in all cases, so the solution to (9′)-(10′) is (up to a constant factor) $Y_n(y) \overset{(1)}{=} \cosh(n\pi y)$ for $n = 0, 1, 2, \ldots$ We have obtained that

(2) $$u_n(x,y) = X_n(x) Y_n(y) = \cos(n\pi x)\cosh(n\pi y)$$

solves (1)-(2)-(3)-(4) for $n = 0, 1, 2, \ldots$ By the superposition principle,

(3) $$u(x,y) = \sum_{n=0}^{N} c_n \cos(n\pi x)\cosh(n\pi y)$$

(25 pts. to here.)

solves (1)-(2)-(3)-(4) for any positive integer N and any choice of constants $c_0, c_1, \ldots, c_N$. We want to choose N and the c_n's in such a way that the nonhomogeneous boundary condition (5) is satisfied; i.e. for all $0 \leq x \leq 1$ we want

(2) $$\frac{1}{2} + \frac{1}{2}\cos(2\pi x) \overset{\text{(given identity)}}{=} \cos^2(\pi x) = u(x,0) = \sum_{n=0}^{N} c_n \cos(n\pi x) = c_0 + c_1 \cos(\pi x) + c_2 \cos(2\pi x) + \ldots$$

By inspection, it suffices to take $\frac{1}{2} \overset{(1)}{=} c_0$, $\frac{1}{2} \overset{(1)}{=} c_2$, and all the other c_n's (1) equal to 0. Therefore

$$\boxed{u(x,y) = \frac{1}{2} + \frac{1}{2}\cos(2\pi x)\cosh(2\pi y)}$$ (5)

solves (1)-(2)-(3)-(4)-(5).

3.(35 pts.) (a) Show that $\Phi=\{\sin(\pi x/2),\sin(3\pi x/2),\sin(5\pi x/2),\dots,\sin((2n-1)\pi x/2),\dots\}$ is an orthogonal set of functions on the interval $(0,1)$. You may find useful the identity

$$\sin(A)\sin(B)=[\cos(A-B)-\cos(A+B)]/2.$$

(b) Compute the Fourier coefficients

$$A_n=\frac{\langle f,\varphi_n\rangle}{\langle \varphi_n,\varphi_n\rangle}\quad (n=1,2,3,\dots)$$

of the function $f(x)=x$ with respect to the orthogonal set Φ on $(0,1)$.

(c) Write the Fourier series of f with respect to Φ on $(0,1)$ and compute the third Fourier partial sum $S_3 f(x)$.

(10 pts.) (a) Let m and n be positive integers. Then $\langle \varphi_m,\varphi_n\rangle=\int_0^1 \varphi_m(x)\overline{\varphi_n(x)}\,dx$

$$=\int_0^1 \sin\left(\frac{(2m-1)\pi x}{2}\right)\sin\left(\frac{(2n-1)\pi x}{2}\right)dx \overset{\text{(given identity)}}{=} \frac{1}{2}\int_0^1\left[\cos\left(\frac{(2m-1)\pi x}{2}-\frac{(2n-1)\pi x}{2}\right)-\cos\left(\frac{(2m-1)\pi x}{2}+\frac{(2n-1)\pi x}{2}\right)\right]$$

$$=\frac{1}{2}\int_0^1\left[\cos((m-n)\pi x)-\cos((m+n-1)\pi x)\right]dx=\begin{cases}\frac{1}{2}\left(x-\frac{1}{\pi(2m-1)}\sin((2m-1)\pi x)\right)\Big|_0^1 & \text{if } m=n,\\[2mm] \frac{1}{2}\left(\frac{\sin((m-n)\pi x)}{(m-n)\pi}-\frac{\sin((m+n-1)\pi x)}{(m+n-1)\pi}\right)\Big|_0^1 & \text{if } m\neq n\end{cases}$$

$$=\begin{cases}1/2 & \text{if } m=n,\\ 0 & \text{if } m\neq n.\end{cases}$$

Therefore $\Phi=\left\{\sin\left(\frac{(2n-1)\pi x}{2}\right)\right\}_{n=1}^{\infty}$ is an orthogonal set on $(0,1)$.

(13 pts.) (b) The n^{th} Fourier coefficient of f with respect to Φ on $(0,1)$ is the number

$$\boxed{A_n}=\frac{\langle f,\varphi_n\rangle}{\langle\varphi_n,\varphi_n\rangle}=\frac{\int_0^1 f(x)\overline{\varphi_n(x)}\,dx}{\int_0^1\varphi_n(x)\overline{\varphi_n(x)}\,dx}=2\int_0^1 \underbrace{x}_{U}\,\underbrace{\sin\left(\frac{(2n-1)\pi x}{2}\right)dx}_{dV}$$

$$=2\left[\cancel{\frac{-2x}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi x}{2}\right)\Big|_0^1}^{\,0}-\int_0^1-\frac{2}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi x}{2}\right)dx\right]=2\left(\frac{2}{(2n-1)\pi}\right)\left(\frac{2}{(2n-1)\pi}\right)\sin\left(\frac{(2n-1)\pi x}{2}\right)\Big|_0^1$$

$$=\frac{8}{(2n-1)^2\pi^2}\sin\left(\frac{(2n-1)\pi}{2}\right)=\boxed{\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2},\ (n=1,2,3,\dots).}$$

(12 pts.) (c) $f(x)\sim\sum_{n=1}^{\infty}A_n\varphi_n(x)=\boxed{\sum_{n=1}^{\infty}\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\sin\left(\frac{(2n-1)\pi x}{2}\right)}$ is the Fourier seris of f withrespect to Φ on $(0,1)$. The third Fourier partial sum of f with respect to Φ on $(0,1)$ is

$$S_3 f(x)=\sum_{n=1}^{3}\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\sin\left(\frac{(2n-1)\pi x}{2}\right)=\boxed{\frac{8}{\pi^2}\left(\sin\left(\frac{\pi x}{2}\right)-\frac{1}{9}\sin\left(\frac{3\pi x}{2}\right)+\frac{1}{25}\sin\left(\frac{5\pi x}{2}\right)\right)}$$

Math 325

Exam III

Summer 2012

n = 16

mean = 78.4

median = 85.5

standard deviation = 25.3

Distribution of Scores

Range	Graduate Grade	Undergraduate Grade	Frequency
87 – 100	A	A	8
73 – 86	B	B	5
60 – 72	C	B	0
50 – 59	C	C	1
0 – 49	F	D	2