Mathematics 325 | Final Exam, Spring 2011 | Name: Dr. Grow

At the end of this exam you will find a list of identities and a brief table of Fourier transforms.

1.(28 pts.) (a) Verify that $u(x,y)=(x+1)y-e^{x}+1$ is a particular solution of the nonhomogeneous partial differential equation $(y+e^{x})u_{x}+(e^{2x}-y^{2})u_{y}=xe^{2x}-xy^{2}$.

(b) Find the general solution of the associated homogeneous equation $(y+e^{x})u_{x}+(e^{2x}-y^{2})u_{y}=0$.

(c) Find the solution of $(y+e^{x})u_{x}+(e^{2x}-y^{2})u_{y}=xe^{2x}-xy^{2}$ that satisfies the auxiliary condition $u(0,y)=y^{2}$ for all real y.

(a) $u_x = y-e^x$ and $u_y = x+1$ so $(y+e^x)u_x + (e^{2x}-y^2)u_y = (y+e^x)(y-e^x)+(e^{2x}-y^2)(x+1) = y^2-e^{2x}+x(e^{2x}-y^2)+e^{2x}-y^2 \overset{\checkmark}{=} xe^{2x}-xy^2$.

(b) Along characteristic curves $\frac{dy}{dx}=\frac{b(x,y)}{a(x,y)}$, the solutions to $a(x,y)u_x+b(x,y)u_y=0$ are constant. In this case $\frac{dy}{dx}=\frac{e^{2x}-y^2}{y+e^x}=\frac{(e^x-y)(e^x+y)}{y+e^x}=e^x-y$ so $\frac{dy}{dx}+y=e^x$. An integrating factor for this linear ODE is $\mu = e^{\int p(x)dx}=e^{\int 1dx}=e^x$. Therefore $e^x\frac{dy}{dx}+ye^x=e^{2x}$ or equivalently $\frac{d}{dx}(e^xy)=e^{2x}$. Integrating gives $e^xy=\frac{1}{2}e^{2x}+c_1$ or $2e^xy-e^{2x}=c$ where $c=2c_1$ is an arbitrary constant. Along such a characterstic curve, a solution $u=u(x,y)$ to $(y+e^x)u_x+(e^{2x}-y^2)u_y=0$ is constant. Therefore along the curve $y=\frac{1}{2}e^x+\frac{c}{2}e^{-x}$,

$$u(x,y)=u\left(x,\tfrac{1}{2}e^x+\tfrac{c}{2}e^{-x}\right)=u\left(0,\tfrac{1}{2}+\tfrac{c}{2}\right)=f(c).$$

Thus $u(x,y)=f(2e^xy-e^{2x})$ is the general solution of $(y+e^x)u_x+(e^{2x}-y^2)u_y=0$, where f is an arbitrary C^1-function of a single real variable.

(c) The general solution of $(y+e^x)u_x+(e^{2x}-y^2)u_y=xe^{2x}-xy^2$ is $u=u_c+u_p$

4 pts to here $= f(2e^xy-e^{2x})+(x+1)y-e^x+1$. We need to choose f so that $u(0,y)=y^2$. I.e. $f(2y-1)+y=y^2$ so $f(2y-1)=y^2-y$. Setting $z=2y-1$, this becomes

8 pts to here $f(z)=\left(\frac{z+1}{2}\right)^2-\frac{z+1}{2}=\frac{z+1}{2}\left[\frac{z+1}{2}-1\right]=\left(\frac{z+1}{2}\right)\left(\frac{z-1}{2}\right)=\frac{z^2-1}{4}$. Thus

12 pts to here $\boxed{u(x,y)=\frac{(2e^xy-e^{2x})^2-1}{4}+(x+1)y-e^x+1}$ is the solution we seek.

Note: This can also be expressed as $u(x,y)=e^{2x}y^2+(x+1-e^{3x})y+\frac{1}{4}e^{4x}-e^x+\frac{3}{4}$.

2.(28 pts.) (a) Classify the second order linear partial differential equation $u_{xx} - 4u_{xt} + 4u_{tt} = 0$ as elliptic, parabolic, or hyperbolic.

(b) Find the general solution of $u_{xx} - 4u_{xt} + 4u_{tt} = 0$ in the xt – plane.

(c) Find the solution of $u_{xx} - 4u_{xt} + 4u_{tt} = 0$ in the xt – plane satisfying $u(x,0) = xe^{2x} + 4x^2$ and $u_t(x,0) = (x-2)e^{2x} + 4x$ for all real x.

(a) $B^2 - 4AC = (-4)^2 - 4(1)(4) = 0$ so the p.d.e. is $\boxed{\text{parabolic}}$.

(b) Note that the p.d.e. can be written $\left(\frac{\partial^2}{\partial x^2} - 4\frac{\partial^2}{\partial x \partial t} + 4\frac{\partial^2}{\partial t^2}\right)u = 0$, or equivalently $\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t}\right)u = 0$. Let $\xi = 2x + t$ and $\eta = x - 2t$. The chain rule implies that as operators $\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = 2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$ and $\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} - 2\frac{\partial}{\partial \eta}$. Therefore $\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t} = 2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - 2\left(\frac{\partial}{\partial \xi} - 2\frac{\partial}{\partial \eta}\right) = 5\frac{\partial}{\partial \eta}$ so the p.de. can be rewritten as $\left(5\frac{\partial}{\partial \eta}\right)\left(5\frac{\partial}{\partial \eta}\right)u = 0$ or $\frac{\partial^2 u}{\partial \eta^2} = 0$. Integrating once gives $\frac{\partial u}{\partial \eta} = c_1(\xi)$ and integrating again gives $u = \int c_1(\xi)\,d\eta = \eta c_1(\xi) + c_2(\xi)$. In xt – coordinates,

$$\boxed{u(x,t) = (x-2t)f(2x+t) + g(2x+t)}$$

where f and g are C^2-functions of a single real variable.

(c) Observe that $u_t(x,t) = -2f(2x+t) + (x-2t)f'(2x+t) + g'(2x+t)$. Therefore the first initial condition gives $xe^{2x} + 4x^2 = u(x,0) \overset{⓪}{=} xf(2x) + g(2x)$. Differentiating this yields

$$e^{2x} + 2xe^{2x} + 8x \overset{①}{=} f(2x) + 2xf'(2x) + 2g'(2x) \quad \text{for all real } x.$$

Applying the second initial condition produces

$$(x-2)e^{2x} + 4x = u_t(x,0) \overset{②}{=} -2f(2x) + xf'(2x) + g'(2x) \quad \text{for all real } x.$$

Multiplying equation ② by -2 and adding the result to equation ① leads to $5e^{2x} = 5f(2x)$ so $f(z) = e^z$ for all real z. Substituting this in equation ⓪ yields $4x^2 = g(2x)$ and thus $g(w) = w^2$ for all real w. Consequently

$$\boxed{u(x,t) = (x-2t)e^{2x+t} + (2x+t)^2}$$

is the solution we seek.

3.(28 pts.) Let $u = u(x,y,z,t)$ denote the temperature at time $t \geq 0$ at each point (x,y,z) of a homogeneous body occupying the spherical region $B = \{(x,y,z) : x^2+y^2+z^2 \leq 25\}$. The body is completely insulated and the initial temperature at each point is equal to its distance from the center of B.

(a) Write (without proof or derivation) the partial differential equation and the complete initial/boundary conditions that govern the temperature function.

(b) Use Gauss' divergence theorem to help show that the heat energy $H(t) = \iiint_B c\rho u(x,y,z,t)\,dxdydz$ of the body at time t is actually a constant function of time. (Here c and ρ denote the constant specific heat and density, respectively, of the material in B.)

(c) Compute the constant steady-state temperature that the body reaches after a long time.

(a)
$$\begin{cases} \dfrac{\partial u}{\partial t} - k\nabla^2 u = 0 & \text{if } x^2+y^2+z^2 < 25,\ t>0, \\ \dfrac{\partial u}{\partial n} = 0 & \text{if } x^2+y^2+z^2 = 25,\ t>0, \\ u(x,y,z,0) = \sqrt{x^2+y^2+z^2} & \text{if } x^2+y^2+z^2<25. \end{cases}$$

(5) (b)
$$\frac{dH}{dt} = \iiint_B \frac{\partial}{\partial t}\left(c\rho u(x,y,z,t)\right)dxdydz = c\rho\iiint_B u_t(x,y,z,t)\,dxdydz =$$
$$kc\rho\iiint_B \nabla^2 u(x,y,z,t)\,dxdydz \overset{\text{Gauss' D.T.}}{=} kc\rho\iint_{\partial B} \nabla u(x,y,z,t)\cdot\vec{n}\,dS = kc\rho\iint_{\partial B} \underbrace{\frac{\partial u}{\partial n}(x,y,z,t)}_{0}\,dS = 0.$$

Therefore H is a constant function of t for $t \geq 0$.

(5) (c) Let $U = \lim_{t\to\infty} u(x,y,z,t)$ be the constant steady-state temperature that the body reaches after a long time. Then

$$c\rho U\,\text{volume}(B) = \iiint_B c\rho U\,dxdydz = \iiint_B c\rho \lim_{t\to\infty} u(x,y,z,t)\,dxdydz = \lim_{t\to\infty}\iiint_B c\rho u(x,y,z,t)\,dxdydz$$
$$= \lim_{t\to\infty} H(t) \overset{\text{(since } H \text{ is constant)}}{=} H(0) = \iiint_B c\rho u(x,y,z,0)\,dxdydz = \iiint_B c\rho\sqrt{x^2+y^2+z^2}\,dxdydz = \int_0^{2\pi}\int_0^{\pi}\int_0^{5} c\rho r\cdot r^2\sin\varphi\,dr\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} c\rho\sin\varphi\left(\int_0^5 r^3\,dr\right)d\varphi\,d\theta = \frac{5^4}{4}c\rho\int_0^{2\pi}\left(\int_0^{\pi}\sin\varphi\,d\varphi\right)d\theta = \frac{5^4}{4}c\rho\int_0^{2\pi}(-\cos\varphi)\Big|_0^{\pi}\,d\theta = 5^4\pi c\rho.$$

Therefore
$$U = \frac{5^4\pi c\rho}{c\rho\,\text{volume}(B)} = \frac{5^4\pi}{\frac{4}{3}\pi(5)^3} = \boxed{\frac{15}{4}} = 3.75.$$

4.(30 pts.) Use Fourier transform methods to solve $u_{xx}+u_{yy}=0$ for $-\infty<x<\infty,\ 0<y<\infty$, subject to the boundary condition

$$u(x,0)=\begin{cases}1 & \text{if } |x|<1,\\ 0 & \text{otherwise,}\end{cases}$$

and the decay condition $\lim_{y\to\infty} u(x,y)=0$ for each x in $(-\infty,\infty)$. (Note: For full credit, do not leave any unevaluated integrals in your final answer.)

Let $u=u(x,y)$ be a solution to the problem. Taking the Fourier transform of $u_{xx}(x,y)+u_{yy}(x,y)=0$ with respect to the variable x we obtain

$$0=\mathcal{F}(0)(\xi)=\mathcal{F}(u_{xx}+u_{yy})(\xi)=\mathcal{F}(u_{xx})(\xi)+\mathcal{F}(u_{yy})(\xi)$$

4 pts. to here.

$$0=\frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi)-\xi^2\mathcal{F}(u)(\xi).$$

7 pts. to here The general solution of this o.d.e. is $\mathcal{F}(u)(\xi)=c_1(\xi)e^{\xi y}+c_2(\xi)e^{-\xi y}$ where c_1 and c_2 are arbitrary functions of a single real variable. Suppose $\xi>0$. Then

$$0=\mathcal{F}(0)(\xi)=\lim_{y\to\infty}\mathcal{F}(u(\cdot,y))(\xi)=\lim_{y\to\infty}\left(c_1(\xi)e^{\xi y}+c_2(\xi)e^{-\xi y}\right)=\lim_{y\to\infty}\left(c_1(\xi)e^{\xi y}\right),$$

10 pts. to here so we must have $c_1(\xi)=0$ for all $\xi>0$. A similar argument shows that $c_2(\xi)=0$ for $\xi<0$. Thus,

13 pts. to here

$$\mathcal{F}(u)(\xi)=\left\{\begin{array}{ll} c_2(\xi)e^{-\xi y} & \text{if } \xi>0,\\ c_1(\xi)e^{\xi y} & \text{if } \xi<0,\end{array}\right\}=A(\xi)e^{-|\xi|y}$$

for $y>0$ and all real ξ. Applying the initial condition, we have

16 pts. to here.

$$A(\xi)=A(\xi)e^{-|\xi|y}\Big|_{y=0}=\mathcal{F}(u)(\xi)\Big|_{y=0}=\mathcal{F}(u(\cdot,0))(\xi)=\hat{\chi}_{(-1,1)}(\xi)$$

where $\chi_{(-1,1)}(x)=\left\{\begin{array}{ll} 1 & \text{if } x\in(-1,1),\\ 0 & \text{otherwise}\,,\end{array}\right\}$ is the characteristic function of the open interval $(-1,1)$. Therefore $\mathcal{F}(u)(\xi) \overset{①}{=} \hat{\chi}_{(-1,1)}(\xi)\, e^{-|\xi| y}$. Using formula C in the Fourier transforms table, we see that

$$\mathcal{F}\left\{\sqrt{\frac{2}{\pi}} \cdot \frac{y}{(\cdot)^2+y^2}\right\}(\xi) = e^{-|\xi| y}$$

for $y>0$. Consequently, substituting in the equation ① and using the convolution theorem

18 pts. to here. $$\mathcal{F}(u)(\xi) = \mathcal{F}\left(\chi_{(-1,1)}\right)(\xi)\, \mathcal{F}\left(\sqrt{\frac{2}{\pi}} \cdot \frac{y}{(\cdot)^2+y^2}\right)(\xi)$$

$$= \frac{1}{\sqrt{2\pi}}\, \mathcal{F}\left(\sqrt{\frac{2}{\pi}}\, \chi_{(-1,1)} * \frac{y}{(\cdot)^2+y^2}\right)(\xi)$$

21 pts. to here. $$= \mathcal{F}\left(\frac{1}{\pi}\chi_{(-1,1)} * \frac{y}{(\cdot)^2+y^2}\right)(\xi).$$

As a consequence of the inversion theorem, it follows that

24 pts. to here. $$u(x,y) = \frac{1}{\pi}\left(\chi_{(-1,1)} * \frac{y}{(\cdot)^2+y^2}\right)(x)$$

for all $-\infty<x<\infty$ and all $y>0$. That is,

27 pts. to here. $$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-s)^2+y^2}\, \chi_{(-1,1)}(s)\, ds = \frac{1}{\pi}\int_{-1}^{1} \frac{y\, ds}{(x-s)^2+y^2} = \frac{1}{\pi}\int_{-1}^{1} \frac{ds/y}{\left(\frac{s-x}{y}\right)^2+1}$$

$$= \frac{1}{\pi}\mathrm{Arctan}\left(\frac{s-x}{y}\right)\Bigg|_{s=-1}^{1} = \frac{1}{\pi}\mathrm{Arctan}\left(\frac{1-x}{y}\right) - \frac{1}{\pi}\mathrm{Arctan}\left(\frac{-1-x}{y}\right)$$

30 pts. to here. $$= \boxed{\frac{1}{\pi}\mathrm{Artan}\left(\frac{x+1}{y}\right) - \frac{1}{\pi}\mathrm{Arctan}\left(\frac{x-1}{y}\right)} \quad \text{for } -\infty<x<\infty \text{ and } y>0.$$

5.(28 pts.) (a) Find a solution to $\nabla^2 u = 0$ in the cube C: $0 < x < 1,\ 0 < y < 1,\ 0 < z < 1$, subject to the boundary conditions $u(x, y, 1) = \sin(\pi x)\sin^3(\pi y)$ for $0 \le x \le 1,\ 0 \le y \le 1$ and $u = 0$ on the other five faces of C.

(b) State the maximum/minimum principle for harmonic functions and use it to show that the problem in part (a) has only one solution.

We must solve $u_{xx} + u_{yy} + u_{zz} \overset{(1)}{=} 0$ in C subject to $u(0,y,z) \overset{(2)}{=} 0 \overset{(3)}{=} u(1,y,z)$ for $0 \le y \le 1,\ 0 \le z \le 1$, $u(x,0,z) \overset{(4)}{=} 0 \overset{(5)}{=} u(x,1,z)$ for $0 \le x \le 1,\ 0 \le z \le 1$, $u(x,y,0) \overset{(6)}{=} 0$ and $u(x,y,1) \overset{(7)}{=} \sin(\pi x)\sin^3(\pi y)$ for $0 \le x \le 1,\ 0 \le y \le 1$. We seek nontrivial solutions of the homogeneous portion of this problem, (1)–(6), of the form $u(x,y,z) = X(x)Y(y)Z(z)$. Substituting in (1) yields $X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0$ in C. Dividing through by $X(x)Y(y)Z(z)$ and rearranging gives

$$\frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = -\frac{X''(x)}{X(x)} = \text{constant} \overset{(8)}{=} \lambda$$

and then

$$\frac{Z''(z)}{Z(z)} - \lambda = -\frac{Y''(y)}{Y(y)} = \text{constant} \overset{(9)}{=} \mu .$$

Substituting $u = XYZ$ in (2) and (3) give $X(0)Y(y)Z(z) = 0 = X(1)Y(y)Z(z)$ for all $0 \le y \le 1, 0 \le z$. In order that the solution u be nontrivial we must have $X(0) \overset{(10)}{=} 0 \overset{(11)}{=} X(1)$. Similar arguments lead to $Y(0) \overset{(12)}{=} 0 \overset{(13)}{=} Y(1)$ and $Z(0) \overset{(14)}{=} 0$ (using (4), (5), and (6), respectively). Thus we are led to the coupled system of ODEs and BCs using (8), (9), (10), (11), (12):

to here

3 to here

13 to here

$$(*) \begin{cases} X''(x) + \lambda X(x) = 0, & X(0) = 0 = X(1), \\ Y''(y) + \mu Y(y) = 0, & Y(0) = 0 = Y(1), \\ Z''(z) - (\lambda + \mu) Z(z) = 0, & Z(0) = 0 . \end{cases}$$

The first two lines of $(*)$ are eigenvalue problems for the operators $-\frac{d^2}{dx^2}$ and $-\frac{d^2}{dy^2}$, respectively, with Dirichlet boundary conditions. The eigenvalues and eigenfunctions are then well-known:

12 to here

14 to here

$$\lambda_\ell = (\ell\pi)^2, \quad X_\ell(x) = \sin(\ell\pi x) \quad (\ell = 1, 2, 3, \dots),$$

$$\mu_m = (m\pi)^2, \quad Y_m(y) = \sin(m\pi y) \quad (m = 1, 2, 3, \dots).$$

Substituting $\lambda = \lambda_\ell$ and $\mu = \mu_m$ in the last line of $(*)$ we find the general solution of the ODE is $Z_{\ell,m}(z) = c_1 \cosh(\pi\sqrt{\ell^2+m^2}\, z) + c_2 \sinh(\pi\sqrt{\ell^2+m^2}\, z)$. Applying the BC $Z_{\ell,m}(0) = 0$ means that $c_1 = 0$ so $Z_{\ell m}(z) = \sinh(\pi z\sqrt{\ell^2+m^2})$, up to a constant multiple.

16 to here

Therefore $u_{\ell,m}(x,y,z) = X_\ell(x)Y_m(y)Z_{\ell,m}(z) = \sin(\ell\pi x)\sin(m\pi y)\sinh\left(\pi z\sqrt{\ell^2+m^2}\right)$

$(\ell = 1,2,3,\dots$ and $m = 1,2,3,\dots)$ solves ①–⑥. By the superposition principle, a formal solution of ①–⑥ is

$$u(x,y,z) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} a_{\ell,m}\sin(\ell\pi x)\sin(m\pi y)\sinh\left(\pi z\sqrt{\ell^2+m^2}\right)$$

for arbitrary constants $a_{\ell,m}$ $(\ell=1,2,3,\dots$ and $m=1,2,3,\dots)$. We need to choose the constants so that ⑦ is satisfied:

$$\sin(\pi x)\sin^3(\pi y) = u(x,y,1) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} a_{\ell,m}\sin(\ell\pi x)\sin(m\pi y)\sinh\left(\pi\sqrt{\ell^2+m^2}\right)$$

for all $0\le x\le 1$, $0\le y\le 1$. Using the identity $\sin^3(A) = \frac{3}{4}\sin(A) - \frac{1}{4}\sin(3A)$ at the end of this exam, this can be rewritten as

$$\frac{3}{4}\sin(\pi x)\sin(\pi y) - \frac{1}{4}\sin(\pi x)\sin(3\pi y) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} a_{\ell,m}\sin(\ell\pi x)\sin(m\pi y)\sinh\left(\pi\sqrt{\ell^2+m^2}\right).$$

By inspection, $\frac{3}{4} = a_{1,1}\sinh(\pi\sqrt{2})$, $-\frac{1}{4} = a_{1,3}\sinh(\pi\sqrt{10})$, and all other $a_{\ell,m} = 0$. Therefore

$$\boxed{u(x,y,z) = \frac{3\sin(\pi x)\sin(\pi y)\sinh(\pi z\sqrt{2})}{4\sinh(\pi\sqrt{2})} - \frac{\sin(\pi x)\sin(3\pi y)\sinh(\pi z\sqrt{10})}{4\sinh(\pi\sqrt{10})}}$$

is a continuous solution of the problem ①–⑦.

(b) Maximum/Minimum Principle: Let $u = u(x,y,z)$ be a solution to Laplace's equation $\nabla^2 u = 0$ in a bounded, open set R of $\mathbb{R}^3$ and let u be continuous on the closure $\overline{R} = R \cup \partial R$ of R. Then

$$\max\{u(x,y,z) : (x,y,z)\in\overline{R}\} = \max\{u(x,y,z) : (x,y,z)\in\partial R\} \text{ and}$$

$$\min\{u(x,y,z) : (x,y,z)\in\overline{R}\} = \min\{u(x,y,z) : (x,y,z)\in\partial R\}.$$

Let $u = u(x,y,z)$ denote the (continuous) solution to the problem ①-⑦ obtained in part (a), and let $u = v(x,y,z)$ be another solution to ①-⑦ that is continuous on $\overline{C} = C \cup \partial C$: $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$. Define w on $\overline{C}$ by setting $w(x,y,z) = u(x,y,z) - v(x,y,z)$. Then w is a continuous function on $\overline{C}$ such that $\nabla^2 w = 0$ in C and $w = 0$ on ∂C. By the maximum/minimum principle for harmonic functions, $w = 0$ on $\overline{C}$. That is, $u(x,y,z) = v(x,y,z)$ for all (x,y,z) in $\overline{C}$. Thus the solution to the problem in part (a) is unique.

6.(28 pts.) The material in a spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution. The material is held at 100 degrees Centigrade on its inner boundary. On its outer boundary, the temperature distribution u of the material satisfies the Neumann condition $\partial u/\partial n = -\gamma$ where γ is a positive constant.

(a) Find the temperature distribution function for the material.

(b) What are the hottest and coldest temperatures in the material?

(c) Is it possible to choose γ so that the temperature on the outer boundary is 20 degrees Centigrade? Support your answer.

(a)
$$\begin{cases} 0 = \nabla^2 u & \text{if } \quad 1 < r < 2, \\ u = 100 & \text{if } \quad r = 1, \\ \dfrac{\partial u}{\partial r} = -\gamma & \text{if } \quad r = 2. \end{cases}$$

We assume a radial solution $u = u(r)$, independent of θ and φ, for this problem based on rotational invariance of the p.d.e. and boundary conditions. Therefore

$$0 = \nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi \frac{\partial u}{\partial \varphi}\right) + \frac{1}{r^2 \sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2}$$

reduces to

$$0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right).$$

Then $r^2 \dfrac{\partial u}{\partial r} = c_1$ so $\dfrac{\partial u}{\partial r} = \dfrac{c_1}{r^2}$. Applying the boundary condition $-\gamma =$

$\dfrac{\partial u}{\partial n} = \dfrac{\partial u}{\partial r}$ when $r = 2$ yields $\dfrac{c_1}{2^2} = -\gamma$ so $c_1 = -4\gamma$.

Integrating $\dfrac{\partial u}{\partial r} = \dfrac{-4\gamma}{r^2}$ leads to $u(r) = \dfrac{4\gamma}{r} + c_2$. Applying the boundary condition $100 = u(1) = \dfrac{4\gamma}{1} + c_2$ yields $c_2 = 100 - 4\gamma$.

Therefore, the temperature distribution function for the material is

$$\boxed{u(r) = \frac{4\gamma}{r} + 100 - 4\gamma}$$

(b) The hottest and coldest temperatures in the material occur on the boundaries of the region.

$$u(1) = \frac{4\gamma}{1} + 100 - 4\gamma = \boxed{100}$$ is the hottest temperature.

$$u(2) = \frac{4\gamma}{2} + 100 - 4\gamma = \boxed{100-2\gamma}$$ is the coldest temperature.

(c) We want $u(2) = 20$. Therefore $\frac{4\gamma}{2} + 100 - 4\gamma = 20$ so $80 = 2\gamma$ and hence $\boxed{\gamma = 40}$. $\boxed{\text{Yes,}}$ it is possible to choose γ so the temperature is 20°C on the outer boundary.

7.(30 pts.) Find a solution of $u_t - t^2 u_{xx} + u \overset{①}{=} 0$ in the open strip $0 < x < 1$, $0 < t < \infty$ such that u is continuous on the closure of the strip, satisfies the boundary conditions $u(0,t) \overset{②}{=} 0 \overset{③}{=} u(1,t)$ for $t \geq 0$ and satisfies the initial condition $u(x,0) \overset{④}{=} 4x(1-x)$ for $0 \leq x \leq 1$.

Bonus (10 pts.): Show that there is at most one solution to this problem.

We seek nontrivial solutions of the homogeneous portion of this problem, ①-②-③, of the form $u(x,t) = X(x)T(t)$. Substituting this expression in ① yields

$$X(x)T'(t) - t^2 X''(x)T(t) + X(x)T(t) = 0.$$

Dividing by $X(x)T(t)$ and rearranging leads to

$$-\frac{X''(x)}{X(x)} = -\frac{1}{t^2}\left(\frac{T'(t)}{T(t)} + 1\right) = \text{constant} = \lambda.$$

Substituting $u = X(x)T(t)$ in ② and ③ produces $X(0)T(t) = 0 = X(1)T(t)$ for $t \geq 0$. In order that $u(x,t) = X(x)T(t)$ be a nontrivial solution in the strip, we must have $X(0) = 0 = X(1)$. Therefore we are lead to the coupled system of ODEs and BCs:

$$(*) \begin{cases} X''(x) + \lambda X(x) = 0, \ X(0) = 0 = X(1), \\ T'(t) + (1 + \lambda t^2)T(t) = 0. \end{cases}$$

The first line of $(*)$ is the eigenvalue problem for the operator $-\frac{d^2}{dx^2}$ with Dirichlet BCs. Thus the eigenvalues and eigenfunctions are well-known: $\lambda_n = (n\pi)^2$ and $X_n(x) = \sin(n\pi x)$ $(n = 1, 2, 3, \ldots)$. Substituting $\lambda = \lambda_n$ in the second line of $(*)$ gives $\frac{T_n'(t)}{T_n(t)} = -(1 + \lambda_n t^2)$, and integrating produces $\ln T_n(t) = -t - \frac{\lambda_n t^3}{3} + c$ so $T_n(t) = A e^{-t} e^{-(n\pi)^2 t^3/3}$ for some constant A. Thus $u_n(x,t) = X_n(x)T_n(t)$ $= \sin(n\pi x) e^{-t} e^{-(n\pi)^2 t^3/3}$, up to a constant multiple. The superposition principle yields the formal solution $u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\pi n x) e^{-t} e^{-n^2\pi^2 t^3/3}$ where the b_ns are arbitrary constants. We need to choose the constants so ④ is satisfied:

$$4x(1-x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \qquad \text{for all } 0 \leq x \leq 1.$$

Therefore we should choose b_n to be the n^{th} Fourier sine coefficient of the function $f(x) = 4x(1-x)$ on $[0,1]$:

$$b_n = \frac{\langle f, \sin(n\pi\cdot)\rangle}{\langle \sin(n\pi\cdot), \sin(n\pi\cdot)\rangle} = \frac{\int_0^1 f(x)\sin(n\pi x)\,dx}{\int_0^1 \sin^2(n\pi x)\,dx} = 2\int_0^1 \overbrace{4x(1-x)}^{U}\overbrace{\sin(n\pi x)dx}^{dV}$$

$$= \cancel{\frac{-8x(1-x)\cos(n\pi x)}{n\pi}}\Bigg|_0^1 \nearrow 0 + 2\int_0^1 \underbrace{(4-8x)}_{U}\overbrace{\left(+\frac{\cos(n\pi x)}{n\pi}\right)dx}^{dV} = \cancel{\frac{2(4-8x)\sin(n\pi x)}{(n\pi)^2}}\Bigg|_0^1 \nearrow 0 + 2\int_0^1 \frac{\sin(n\pi x)}{(n\pi)^2}(+8\,dx$$

$$= \frac{-16\cos(n\pi x)}{(n\pi)^3}\Bigg|_0^1 = \frac{-16\left((-1)^n - 1\right)}{(n\pi)^3} = \begin{cases} 0 & \text{if } n = 2k \text{ is even,} \\ \dfrac{32}{\pi^3(2k+1)^3} & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

Because $f(x) = 4x(1-x)$ is twice continuously differentiable on $[0,1]$ and f satisfies the Dirichlet boundary conditions $\varphi(0) = 0 = \varphi(1)$ that give rise to the eigenfunctions $\left\{\sin(n\pi x)\right\}_{n=1}^{\infty}$ of the operator $-\frac{d^2}{dx^2}$ on $[0,1]$, the Fourier sine series of f converges uniformly to f on $[0,1]$:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = \sum_{k=0}^{\infty} \frac{32\sin((2k+1)\pi x)}{\pi^3(2k+1)^3} \qquad \text{for all } 0 \le x \le 1.$$

Therefore

$$\boxed{u(x,t) = e^{-t}\sum_{k=0}^{\infty} \frac{32\sin((2k+1)\pi x)\,e^{-\frac{(2k+1)^3\pi^2 t^2}{3}}}{\pi^3(2k+1)^3}}$$

is a (continuous) solution to ①-②-③-④.

Bonus: Let $u = v(x,t)$ be another solution to ①-②-③-④ which is continuous on the closed strip: $0 \le x \le 1$, $0 \le t < \infty$. Consider the function w defined on the strip by

$$w(x,t) = u(x,t) - v(x,t),$$

and let

$$\text{Ⓗ}(t) = \left(\int_0^1 w^2(x,t)\,dx \right)^{1/2}$$

be the root mean-square "energy" of w at time $t \ge 0$. Then differentiating we find

$$2\,\text{Ⓗ}(t)\,\text{Ⓗ}'(t) \overset{⑤}{=} \frac{d}{dt}\int_0^1 w^2(x,t)dx = \int_0^1 \frac{\partial}{\partial t}\left(w^2(x,t)\right)dx = \int_0^1 2w(x,t)w_t(x,t)dx.$$

Since w satisfies

$$\begin{cases} w_t - t^2 w_{xx} + w \overset{⑥}{=} 0 & \text{if } 0<x<1,\ 0<t<\infty, \\ w(0,t) \overset{⑦}{=} 0 \overset{⑧}{=} w(1,t) & \text{if } t \ge 0, \\ w(x,0) \overset{⑨}{=} 0 & \text{if } 0 \le x \le 1, \end{cases}$$

substituting from ⑥ in ⑤ yields

$$\begin{aligned} \text{Ⓗ}(t)\,\text{Ⓗ}'(t) &= \int_0^1 w(x,t)\left[t^2 w_{xx}(x,t) - w(x,t)\right]dx \\ &= -\int_0^1 w^2(x,t)dx + t^2\int_0^1 \overbrace{w(x,t)}^{v}\,\overbrace{w_{xx}(x,t)\,dx}^{dV} \\ &= -\int_0^1 w^2(x,t)dx + t^2 w(x,t)w_x(x,t)\Big|_{x=0}^{1} - t^2\int_0^1 w_x^2(x,t)dx. \end{aligned}$$

But ⑦ and ⑧ imply $t^2 w(x,t)w_x(x,t)\Big|_{x=0}^{1} = 0$ so $\text{Ⓗ}(t)\text{Ⓗ}'(t) = -\int_0^1 w^2dx - t^2\int_0^1 w_x^2 dx$ ≤ 0 for $t \ge 0$. Since $\text{Ⓗ}(t) \ge 0$ it follows that $\text{Ⓗ}'(t) \le 0$; i.e. Ⓗ is a decreasing function on $t \ge 0$. Thus $0 \le \text{Ⓗ}(t) \le \text{Ⓗ}(0) = \left(\int_0^1 w^2(x,0)dx\right)^{1/2} = 0$ by ⑨, and consequently $\text{Ⓗ}(t) = 0$ for $t \ge 0$. Therefore $w^2(x,t) = 0$ for all $0 \le x \le 1$ and each $t \ge 0$. Hence $u(x,t) = v(x,t)$ for $0 \le x \le 1$, $0 \le t < \infty$. I.e. there is only one continuous solution to the problem ①-②-③-④.

Trigonometric Identities

$$\cos^2(A) = \frac{1}{2} + \frac{1}{2}\cos(2A)$$

$$\sin^2(A) = \frac{1}{2} - \frac{1}{2}\cos(2A)$$

$$\sin(A)\sin(B) = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$$

$$\sin(A)\cos(B) = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)$$

$$\cos(A)\cos(B) = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$$

$$\cos^3(A) = \frac{3}{4}\cos(A) + \frac{1}{4}\cos(3A)$$

$$\sin^3(A) = \frac{3}{4}\sin(A) - \frac{1}{4}\sin(3A)$$

Expressions for the Laplacian Operator

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad \text{(polar coordinates)}$$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \qquad \text{(cylindrical coordinates)}$$

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin(\varphi)}\frac{\partial}{\partial \varphi}\left(\sin(\varphi)\frac{\partial u}{\partial \varphi}\right) + \frac{1}{r^2\sin^2(\varphi)}\frac{\partial^2 u}{\partial \theta^2} \qquad \text{(spherical coordinates)}$$

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\,dx$
A.	$\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{1}{(a + i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b(\xi - a))}{\xi - a}$
H.	$\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{i(\xi - a)\sqrt{2\pi}}$
I.	$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x} \quad (a > 0)$	$\begin{cases} 0 & \text{if } \lvert\xi\rvert \ge a, \\ \sqrt{\frac{\pi}{2}} & \text{if } \lvert\xi\rvert < a. \end{cases}$

Math 325
Final Exam
Spring 2011

mean: 118.5

median: 125

standard deviation: 53.5

number taking exam: 21

Distribution of Scores:

	Graduate Letter Grade	Undergrad. Letter Grade	Frequency
174 – 200	A	A	5
146 – 173	B	B	1
120 – 145	C	B	5
100 – 119	C	C	4
0 – 99	F	D	6