Mathematics 325 | Final Exam Summer 2011 | Name: Dr. Grow

The first two pages of this exam contain **eight** problems which are of equal value ... 25 points each. The last two pages of this exam consist of a table of Fourier transforms and some Fourier series convergence theorems. Furthermore, you may find one or more of the following identities useful on this exam.

$$\cos^2(\theta)=\frac{1}{2}+\frac{1}{2}\cos(2\theta)$$

$$\sin^3(\theta)=\frac{3}{4}\sin(\theta)-\frac{1}{4}\sin(3\theta)$$

$$\nabla^2 u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$\nabla^2 u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin(\varphi)}\frac{\partial}{\partial\varphi}\left(\sin(\varphi)\frac{\partial u}{\partial\varphi}\right)+\frac{1}{r^2\sin^2(\varphi)}\frac{\partial^2 u}{\partial\theta^2}$$

1.(25 pts.) Solve $\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x\partial y}-20\frac{\partial^2 u}{\partial x^2}=0$ subject to $u(0,y)=25y^2+64y^3$ and $\frac{\partial u}{\partial x}(0,y)=-10y+48y^2$ for all real y.

2.(25 pts.) Solve $\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial y^2}=0$ in $0<x<\infty$, $-\infty<y<\infty$, subject to $u(0,y)=e^{3y}$ for all real y.

3.(25 pts.) (a) Solve $\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial y^2}\overset{1}{=}0$ in $-1<y<1$, $0<x<\infty$, subject to $u(x,-1)\overset{2}{=}u(x,1)$ and $\frac{\partial u}{\partial y}(x,-1)\overset{3}{=}\frac{\partial u}{\partial y}(x,1)$ for $x\ge 0$ and $u(0,y)\overset{4}{=}\cos^2(\pi y)$ for $-1\le y\le 1$.

(b) Show that the solution to the problem in part (a) is unique.

4.(25 pts.) (a) Show that the Fourier cosine series of the function $f(y)=y^4-2y^2$ on the interval $0\le y\le 1$ is

$$\frac{-7}{15}+\frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos(n\pi y)}{n^4}$$

(b) Discuss the convergence or lack thereof for the Fourier cosine series of f on $0\le y\le 1$. Be sure to give reasons for your answers for all three types of convergence: uniform, L^2, and pointwise.

5.(25 pts.) Use Fourier transform methods to solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ in $-\infty<x<\infty$, $0<y<\infty$, subject to

$$u(x,0)=\begin{cases}1 & \text{if } 0<x<1,\\ 0 & \text{otherwise,}\end{cases}$$

and $\lim_{y\to\infty}u(x,y)=0$ for each real number x. Note: For full credit, do not leave any unevaluated integrals in your final answer.

6.(25 pts.) Solve $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^4 u}{\partial y^4}=0$ in $0<x<\infty,\ 0<y<1$, subject to $\frac{\partial u}{\partial y}(x,0)=0=\frac{\partial u}{\partial y}(x,1)$ and $\frac{\partial^3 u}{\partial y^3}(x,0)=0=\frac{\partial^3 u}{\partial y^3}(x,1)$ for $x\geq 0$ and $u(0,y)=y^4-2y^2$ and $\frac{\partial u}{\partial x}(0,y)=0$ for $0\leq y\leq 1$. Note: You may find the results of problem 4 useful.

Bonus (10 pts.): Show that there is at most one solution to the above problem.

7.(25 pts.) Solve $\nabla^2 u=1$ in the spherical shell $1<x^2+y^2+z^2<4$ subject to the conditions that u vanishes on the inner boundary, i.e. $u=0$ if $x^2+y^2+z^2=1$, and the normal derivative of u vanishes on the outer boundary, i.e. $\frac{\partial u}{\partial n}=0$ if $x^2+y^2+z^2=4$.

8.(25 pts.) (a) Solve $\nabla^2 u=0$ in the cube $0<x<\pi,\ 0<y<\pi,\ 0<z<\pi$, given that $u(x,y,\pi)=\sin(x)\sin^3(y)$ if $0\leq x\leq\pi,\ 0\leq y\leq\pi$, and that u satisfies homogeneous Dirichlet boundary conditions on the other five faces of the cube.

(b) State the maximum-minimum principle for harmonic functions and use it to show that there is at most one solution to the problem in part (a).

Fourier Series Convergence Theorems

Consider the eigenvalue problem

(1) $$X''(x)+\lambda X(x)=0 \text{ in } a<x<b$$

with any symmetric boundary conditions of the form

(2) $$\begin{cases} \alpha_1 f(a)+\beta_1 f(b)+\gamma_1 f'(a)+\delta_1 f'(b)=0 \\ \alpha_2 f(a)+\beta_2 f(b)+\gamma_2 f'(a)+\delta_2 f'(b)=0 \end{cases}$$

and let $\Phi=\{X_1,X_2,X_3,\ldots\}$ be the complete orthogonal set of eigenfunctions for (1)-(2). Let f be any absolutely integrable function defined on $a\le x\le b$. Consider the Fourier series for f with respect to Φ:

$$\sum_{n=1}^{\infty} A_n X_n(x)$$

where

$$A_n=\frac{\langle f,X_n\rangle}{\langle X_n,X_n\rangle} \qquad (n=1,2,3,\ldots).$$

Theorem 2. (Uniform Convergence) If

(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a\le x\le b$ and

(ii) f satisfies the given symmetric boundary conditions,

then the Fourier series of f converges uniformly to f on $[a,b]$.

Theorem 3. (L^2 –Convergence) If

$$\int^{b} |f(x)|^2\,dx<\infty$$

then the Fourier series of f converges to f in the mean-square sense in (a,b).

Theorem 4. (Pointwise Convergence of Classical Fourier Series)

(i) If f is a continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) at x converges pointwise to $f(x)$ in the open interval $a<x<b$.

(ii) If f is a piecewise continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) converges pointwise at every point x in $(-\infty,\infty)$. The sum of the Fourier series is

$$\sum_{n=1}^{\infty} A_n X_n(x)=\frac{f(x^+)+f(x^-)}{2}$$

for all x in the open interval (a,b).

Theorem 4∞. If f is a function of period $2l$ on the real line for which f and f' are piecewise continuous, then the classical full Fourier series converges to $\dfrac{f(x^+)+f(x^-)}{2}$ for every real x.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$
A.	$\begin{cases} 1 & \text{if } -b<x<b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c<x<d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi}-e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2+a^2} \quad (a>0)$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases} x & \text{if } 0<x\le b, \\ 2b-x & \text{if } b<x<2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1+2e^{-ib\xi}-e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x>0, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{1}{(a+i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b<x<c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\xi)c}-e^{(a-i\xi)b}}{(a-i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b<x<b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b(\xi-a))}{\xi-a}$
H.	$\begin{cases} e^{iax} & \text{if } c<x<d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{ic(a-\xi)}-e^{id(a-\xi)}}{i(\xi-a)\sqrt{2\pi}}$
I.	$e^{-ax^2} \quad (a>0)$	$\frac{1}{\sqrt{2a}}e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x} \quad (a>0)$	$\begin{cases} 0 & \text{if } \lvert\xi\rvert\ge a, \\ \sqrt{\frac{\pi}{2}} & \text{if } \lvert\xi\rvert<a. \end{cases}$

$\underline{\#1}$ $\quad \left(\frac{\partial}{\partial y}+5\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial y}-4\frac{\partial}{\partial x}\right)u=0.$

Let $\begin{cases} \xi = 5y - x \\ \eta = +4y + x. \end{cases}$ Then $\frac{\partial}{\partial y} = \overbrace{\frac{\partial \xi}{\partial y}}^{5}\cdot\frac{\partial}{\partial \xi} + \overbrace{\frac{\partial \eta}{\partial y}}^{4}\cdot\frac{\partial}{\partial \eta} = 5\frac{\partial}{\partial \xi} + 4\frac{\partial}{\partial \eta}$

3pts. to here.

and $\frac{\partial}{\partial x} = \overbrace{\frac{\partial \xi}{\partial x}}^{-1}\cdot\frac{\partial}{\partial \xi} + \overbrace{\frac{\partial \eta}{\partial x}}^{+1}\cdot\frac{\partial}{\partial \eta} = -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}.$

6pts. to here

The PDE therefore becomes

$$\left[\left(\cancel{5\frac{\partial}{\partial \xi}}+4\frac{\partial}{\partial \eta}\right)+5\left(\cancel{-\frac{\partial}{\partial \xi}}+\frac{\partial}{\partial \eta}\right)\right]\left[\left(5\frac{\partial}{\partial \xi}+\cancel{4\frac{\partial}{\partial \eta}}\right)-4\left(-\frac{\partial}{\partial \xi}+\cancel{\frac{\partial}{\partial \eta}}\right)\right]u=0$$

$$\left(25\frac{\partial}{\partial \eta}\right)\left(9\frac{\partial}{\partial \xi}\right)u=0$$

$$\frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi}\right)=0.$$

Integrating once gives $\frac{\partial u}{\partial \xi} = c_1(\xi)$ and integrating again gives $u = \int c_1(\xi)\,d\xi + c_2(\eta) = f(\xi)+g(\eta)$ where f and g are $\overbrace{\ }^{\text{any}}$ C^2-functions of a single real variable. Consequently, the general solution of the PDE is

$$u(x,y) = f(5y-x)+g(4y+x).$$

Note that $\frac{\partial u}{\partial x} = -f'(5y-x)+g'(4y+x)$. We need to choose f and g so that

$$(*) \qquad 25y^2+64y^3 = u(0,y) = f(5y)+g(4y)$$

and

15 pts. to here.

$$(**) \qquad -10y+48y^2 = \frac{\partial u}{\partial x}(0,y) = -f'(5y)+g'(4y)$$

for all real y. Differentiating $(*)$ yields

(***) $50y + 192y^2 = 5f'(5y) + 4g'(4y)$.

Adding 5 times equation (**) to equation (***) leads to

$$432y^2 = 9g'(4y)$$

$$48y^2 = g'(4y)$$

$$3(4y)^2 = g'(4y)$$

$$3z^2 = g'(z).$$

Consequently, an integration yields $g(z) = z^3 + c$ for all real z. 19 pts. to here.

Substituting this into (*) produces

$$25y^2 + 64y^3 = f(5y) + (4y)^3 + c$$

so $$25y^2 = f(5y) + c$$

$$(5y)^2 = f(5y) + c$$

and hence $f(z) = z^2 - c$ for all real z.

It follows from

$$u(x,y) = f(5y-x) + g(4y+x)$$

that

$$u(x,y) = (5y-x)^2 - c + (4y+x)^3 + c$$

or

$$\boxed{u(x,y) = (5y-x)^2 + (4y+x)^3}$$

#2 A solution to $u_t - ku_{xx} = 0$ in $-\infty < x < \infty$, $0 < t < \infty$, satisfying $u(x,0) = \varphi(x)$ for $-\infty < x < \infty$ is

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy,$$

at least for continuous, bounded functions φ. Therefore

6 $$u(x,y) = \frac{1}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} e^{-\frac{(y-\tau)^2}{4x}} \cdot e^{3\tau}\,d\tau \qquad [(x,t) \leftrightarrow (y,x)]$$

is a candidate for a solution to our problem. (Note that $\varphi(\tau) = e^{3\tau}$ is continuous but not bounded on $-\infty < \tau < \infty$, so we will need to check our "solution" at the end of the problem.) Then

9 $$u(x,y) = \frac{1}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} e^{-\left(\frac{y^2 - 2\tau y + \tau^2 - 12\tau x}{4x}\right)}\,d\tau.$$

Completing the square in τ in the exponent of the integrand, we have

$$\begin{aligned} y^2 - 2\tau y + \tau^2 - 12\tau x &= \tau^2 - 2\tau(6x+y) + (6x+y)^2 - (6x+y)^2 + y^2 \\ &= (\tau - (6x+y))^2 - (36x^2 + 12xy + \cancel{y^2}) + \cancel{y^2} \\ &= (\tau - 6x - y)^2 - 4x(9x+3y). \end{aligned}$$ 14

Thus

18 $$u(x,y) = \frac{1}{\sqrt{4\pi x}} \int_{-\infty}^{\infty} e^{-\frac{(\tau-6x-y)^2}{4x}} \cdot e^{9x+3y}\,d\tau.$$ Let $p = \dfrac{\tau - 6x - y}{\sqrt{4x}}$

Then $dp = \dfrac{d\tau}{\sqrt{4x}}$, as $\tau \to +\infty$, $p \to +\infty$, and as τ $-\infty$ so does p. Therefore

24 $$u(x,y) = \frac{e^{9x+3y}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2}\,dp = \boxed{e^{9x+3y}}.$$

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25 (Note that $u_x - u_{yy} = 9e^{9x+3y} - 9e^{9x+3y} \overset{\checkmark}{=} 0$ and $u(0,y) \overset{\checkmark}{=} e^{3y}$.)

#3 (a) We use separation of variables. We seek nontrivial solutions to the homogeneous portion of the problem, ①-②-③, of the form $u(x,y) \overset{(*)}{=} X(x)Y(y)$. Substituting $(*)$ into ① yields $X'(x)Y(y) - X(x)Y''(y) = 0$ or $-\dfrac{X'(x)}{X(x)} = -\dfrac{Y''(y)}{Y(y)} = \text{constant} = \lambda$.
Substituting $(*)$ into ② leads to $X(x)\left[Y(1) - Y(-1)\right] = 0$ for all $x \geq 0$ so we must have $Y(1) = Y(-1)$ for nontrivial solutions. Similarly, substituting $(*)$ into ③ leads to $Y'(1) = Y'(-1)$. Therefore we must solve the coupled system

Eigenvalue Problem

$$\boxed{Y''(y) + \lambda Y(y) = 0, \quad Y(1) = Y(-1), \quad Y'(1) = Y'(-1)},$$

$$X'(x) + \lambda X(x) = 0.$$

In the course, we learned that the eigenvalues and eigenfunctions for $T = -\dfrac{d^2}{dy^2}$ with periodic boundary conditions on $(-l, l) = (-1, 1)$ are:

$$\lambda_n = (n\pi)^2 \quad (n = 0, 1, 2, \dots),$$

$$Y_0(y) = 1 \text{ and } Y_n(y) = a_n \cos(n\pi y) + b_n \sin(n\pi y) \quad (n = 1, 2, 3, \dots),$$

(arbitrary constants)

respectively. The corresponding equation in x becomes

$$X_n'(x) + (n\pi)^2 X_n(x) = 0$$

and the general solution is $X_n(x) = c_n e^{-(n\pi)^2 x}$. Thus

$$u_0(x,y) = X_0(x) Y_0(y) = 1$$

$$u_n(x,y) = X_n(x) Y_n(y) = e^{-(n\pi)^2 x}\left[a_n \cos(n\pi y) + b_n \sin(n\pi y)\right] \quad (n = 1, 2, 3, \dots)$$

are solutions to ①-②-③. The superposition principle then shows that

$$u(x,y) = a_0 + \sum_{n=1}^{N} e^{-(n\pi)^2 x}\left[a_n \cos(n\pi y) + b_n \sin(n\pi y)\right]$$

solves ①-②-③ for any integer $N \geq 1$ and any choice of constants $a_0, a_1, b_1, \dots, a_N, b_N$. We need to choose N and the a_n, b_n's so that ④ is satisfied. That is, so that

$$\frac{1}{2}+\frac{1}{2}\cos(2\pi y) = \cos^2(\pi y) = u(0,y) = a_0 + \sum_{n=1}^{N}\left[a_n\cos(n\pi y)+b_n\sin(n\pi y)\right]$$

for all y in $[-1,1]$. By inspection, we see that $N=2$ and

18 $a_0=\frac{1}{2}$, $a_1=b_1=0$, $a_2=\frac{1}{2}$, and $b_2=0$ suffice. Thus

19 $$\boxed{u(x,y) = \frac{1}{2}+\frac{1}{2}\cos(2\pi y)e^{-4\pi^2 x}}$$

solves ①-②-③-④.

(b) Let $v=v(x,y)$ be another solution to ①-②-③-④ and consider the function $W(x,y)=u(x,y)-v(x,y)$ for $-1\le y\le 1$, $0\le x<\infty$. Then W solves

$$\begin{cases} W_x - W_{yy} \overset{⑤}{=} 0 & \text{if } -1<y<1,\ 0<x<\infty, \\ W(x,-1) \overset{⑥}{=} W(x,1) \text{ and } W_y(x,-1) \overset{⑦}{=} W_y(x,1) & \text{if } x\ge 0 \\ W(0,y) \overset{⑧}{=} 0 & \text{if } -1\le y\le 1. \end{cases}$$

Let $E(x)=\left\{\int_{-1}^{1}\left(W(x,y)\right)^2 dy\right\}^{1/2}$ denote the root-mean square "temperature" of the solution W at $x\ge 0$. Then $\frac{dE^2}{dx} = \int_{-1}^{1}\frac{\partial}{\partial x}W^2(x,y)\,dy = \int_{-1}^{1}2W(x,y)W_x(x,y)\,dy$

$$\overset{⑤}{=} \int_{-1}^{1}\overbrace{2W(x,y)}^{U}\overbrace{W_{yy}(x,y)dy}^{dV} \overset{\text{integration by parts}}{=} 2W(x,y)W_y(x,y)\Big|_{y=-1}^{1} - 2\int_{-1}^{1}W_y^2(x,y)\,dy$$

$$= 2\Big[\overbrace{W(x,1)W_y(x,1)-W(x,-1)W_y(x,-1)}^{0 \text{ by ⑥ and ⑦}}\Big] - 2\int_{-1}^{1}W_y^2(x,y)\,dy \le 0.$$ Thus

$E=E(x)$ is a decreasing function on $x\ge 0$, so $0\le E(x)\le E(0)\overset{\text{by ⑧}}{=}0$.

Consequently $\underbrace{}_{\text{the vanishing theorem implies}}$ $W(x,y)=u(x,y)-v(x,y)=0$ for all $-1\le y\le 1$ and $0\le x<\infty$. I.e. the solution is unique to the problem in part (a).

#4 (a) $f(y) = y^4 - 2y^2 \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi y)$ on $0 \le y \le 1$

where $a_0 = \int_0^1 f(y)\,dy = \left(\frac{1}{5}y^5 - \frac{2}{3}y^3\right)\Big|_0^1 = \frac{-7}{15}$ and, for $n \ge 1$,

$a_n = 2\int_0^1 f(y)\cos(n\pi y)\,dy$. Write $\varphi^{(4)}(y) = \cos(n\pi y)$. Then integrating by parts four times gives

$$\int_0^1 f\varphi^{(4)}dy = \left(f\varphi^{(3)} - f'\varphi'' + f''\varphi' - f^{(3)}\varphi\right)\Big|_0^1 + \int_0^1 f^{(4)}\varphi\,dy.$$

But $\varphi^{(3)}(y) = \frac{\sin(n\pi y)}{n\pi}$ so $\varphi^{(3)}(0) = 0 = \varphi^{(3)}(1)$, $f'(y) = 4y^3 - 4y$ so $f'(0) = 0 = f'(1)$, $\varphi'(y) = \frac{-\sin(n\pi y)}{(n\pi)^3}$ so $\varphi'(0) = 0 = \varphi'(1)$, $f^{(3)}(y) = 24y$ so $f^{(3)}(0) = 0$, and $f^{(4)}(y) = 24$, $\varphi(y) = \frac{\cos(n\pi y)}{(n\pi)^4}$, so $\left(f\varphi^{(3)} - f'\varphi'' + f''\varphi' - f^{(3)}\varphi\right)\Big|_0^1 = -f^{(3)}(1)\varphi(1)$ and $\int_0^1 f^{(4)}\varphi(y)\,dy = \frac{24}{(n\pi)^4}\int_0^1 \cos(n\pi y) = 0$. Therefore

$$a_n = 2\int_0^1 f(y)\cos(n\pi y)\,dy = 2\int_0^1 f(y)\varphi^{(4)}(y)\,dy = -2f^{(3)}(1)\varphi(1) = \frac{-2(24)(-1)^n}{(n\pi)^4} = \frac{48(-1)^{n+1}}{\pi^4 n^4}.$$

Consequently, the Fourier cosine series for f on $[0,1]$ is

$$f(y) \sim -\frac{7}{15} + \frac{48}{\pi^4}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cos(n\pi y)}{n^4}.$$

(b) We use the uniform convergence result (Theorem 2). Note that $f(y) = y^4 - 2y^2$, $f'(y) = 4y^3 - 4y$, and $f''(y) = 12y^2 - 4$ so $f, f',$ and f'' are continuous on $[0,1]$. Furthermore $f'(0) = 0 = f'(1)$ so f satisfies the symmetric homogeneous boundary conditions, $\varphi'(0) = 0$ and $\varphi'(1) = 0$, for $T = \frac{-d^2}{dy^2}$ on $[0,1]$ which generates the orthogonal set $\Phi = \left\{ \cos(n\pi y) \right\}_{n=0}^{\infty}$.

Thus, by Theorem 2, the Fourier cosine series for f on $[0,1]$ converges uniformly to f on $[0,1]$. Since uniform convergence (at least on bounded intervals) implies L^2 and pointwise convergence, it follows that the Fourier cosine series for f converges to f in both the L^2-sense and in the pointwise sense.

#5. Suppose that $u=u(x,y)$ is a solution to this problem. Then $u_{xx}(x,y)+u_{yy}(x,y)\overset{①}{=}0$ for all $-\infty<x<\infty$, $0<y<\infty$, $u(x,0)\overset{②}{=}\varphi(x)=\begin{cases}1 & \text{if } 0<x<1,\\ 0 & \text{o.w.,}\end{cases}$ and $\lim_{y\to\infty} u(x,y)\overset{③}{=}0$ for all $-\infty<x<\infty$. We take the Fourier transform of ① with respect to x: $\mathcal{F}(u_{xx})(\xi)+\mathcal{F}(u_{yy})(\xi)=\mathcal{F}(0)(\xi)=0$. But $\mathcal{F}(f')(\xi)= i\xi\mathcal{F}(f)(\xi)$ and $\mathcal{F}(u_{yy})(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u_{yy}(x,y)e^{-ix\xi}dx = \frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,y)e^{-ix\xi}dx\right)=\frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi)$

and it follows that

$$(-i\xi)^2\mathcal{F}(u)(\xi)+\frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi)=0$$

$$\frac{\partial^2}{\partial y^2}\mathcal{F}(u)(\xi)-\xi^2\mathcal{F}(u)(\xi)=0.$$

The general solution of this second-order ODE in y (with parameter ξ) is $\mathcal{F}(u)(\xi)=c_1(\xi)e^{\xi y}+c_2(\xi)e^{-\xi y}$. Applying ③ we have

$$0=\lim_{y\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,y)e^{-i\xi x}dx=\lim_{y\to\infty}\mathcal{F}(u)(\xi)=\lim_{y\to\infty}\left(c_1(\xi)e^{\xi y}+c_2(\xi)e^{-\xi y}\right).$$

Consequently, $c_1(\xi)=0$ if $\xi>0$ and $c_2(\xi)=0$ if $\xi<0$, so

$$\mathcal{F}(u)(\xi)=\left\{\begin{matrix} c_2(\xi)e^{-\xi y} & \text{if } \xi>0\\ c_1(\xi)e^{\xi y} & \text{if } \xi<0\end{matrix}\right\}=A(\xi)e^{-|\xi|y}.$$

Applying ② leads to

13 $$\mathcal{F}(\varphi)(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,0)e^{-i\xi x}dx=\mathcal{F}(u)(\xi)\Big|_{y=0}=A(\xi)e^{-|\xi|y}\Big|_{y=0}=A(\xi).$$

Thus, entry C in the table of Fourier transforms (with $a=y$) and the convolution theorem imply

$$\mathcal{F}(u)(\xi)=\mathcal{F}(\varphi)(\xi)e^{-|\xi|y}=\mathcal{F}(\varphi)(\xi)\mathcal{F}\left(\sqrt{\tfrac{2}{\pi}}\frac{y}{(\cdot)^2+y^2}\right)(\xi)=\frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\sqrt{\tfrac{2}{\pi}}\frac{y}{(\cdot)^2+y^2}*\varphi\right)(\xi)$$

18 $$=\mathcal{F}\left(\frac{1}{\pi}\frac{y}{(\cdot)^2+y^2}*\varphi\right)(\xi).$$

Therefore, the inversion theorem implies that for $-\infty < x < \infty$ and $y > 0$,

20 $$u(x,y) = \frac{1}{\pi}\left(\frac{y}{(\cdot)^2+y^2} * \varphi\right)(x)$$

22 $$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-z)^2+y^2}\,\varphi(z)\,dz$$

$$= \frac{1}{\pi}\int_{0}^{1} \frac{y}{(x-z)^2+y^2}\,dz$$

$$= \frac{1}{\pi}\int_{0}^{1} \frac{1}{\left(\frac{x-z}{y}\right)^2+1}\cdot\frac{dz}{y}$$

Let $w = \frac{z-x}{y}$. Then $dw = \frac{dz}{y}$, $z=0 \Rightarrow w = -\frac{x}{y}$ and $z=1 \Rightarrow w = \frac{1-x}{y}$.

$$= \frac{1}{\pi}\int_{-\frac{x}{y}}^{\frac{1-x}{y}} \frac{1}{w^2+1}\,dw$$

25 $$= \boxed{\frac{1}{\pi}\operatorname{Arctan}\left(\frac{1-x}{y}\right) - \frac{1}{\pi}\operatorname{Arctan}\left(\frac{-x}{y}\right)}.$$

\# 6 We use the method of separation of variables. We seek nontrivial solutions of the homogeneous portion of the problem, ①-②-③-④-⑤-⑥, of the form $u(x,y) \overset{\circledast}{=} X(x)Y(y)$. Substituting ⊛ in ① gives $X''(x)Y(y) + X(x)Y^{(4)}(y) = 0$ or

$$\frac{X''(x)}{X(x)} = -\frac{Y^{(4)}(y)}{Y(y)} = \text{constant} = \lambda.$$

Substituting ⊛ in ② yields $X(x)Y'(0) = 0$ for $x \geq 0$, so for nontrivial solutions to exist we must have $Y'(0) = 0$. Similarly, substituting ⊛ in ③, ④, ⑤, and ⑥ leads to the conditions $Y'(1) = 0$, $Y^{(3)}(0) = 0 = Y^{(3)}(1)$, and $X'(0) = 0$ for the existence of nontrivial solutions. Hence we have the coupled system:

Eigenvalue Problem

$$\begin{cases} \boxed{Y^{(4)}(y) + \lambda Y(y) = 0, \quad Y'(0) = 0 = Y'(1), \quad Y^{(3)}(0) = 0 = Y^{(3)}(1),} \\ X''(x) - \lambda X(x) = 0, \quad X'(0) = 0. \end{cases}$$

On Exam III, we learned that $T = -\frac{d^4}{dy^4}$ is symmetric on the subspace

$$V = \left\{ \varphi \in C^4[0,1] : \varphi'(0) = 0 = \varphi'(1),\ \varphi^{(3)}(0) = 0 = \varphi^{(3)}(1) \right\}$$

and has eigenvalues $\lambda_n = -(n\pi)^4$ and corresponding eigenfunctions $\varphi_n(y) = \cos(n\pi y)$ where $n = 0, 1, 2, \ldots$

The corresponding problem in x is then

$$X_n''(x) + n^4\pi^4 X_n(x) = 0, \quad X_n'(0) = 0,$$

with solution $X_n(x) = \cos(n^2\pi^2 x)$, up to a constant factor. Therefore

$$u_n(x,y) = X_n(x)Y_n(y) = \cos(n\pi y)\cos(n^2\pi^2 x) \qquad (n = 0, 1, 2, \ldots)$$

solves ①-②-③-④-⑤-⑥, and by the superposition principle the same is true of the formal series

$$u(x,y) = \sum_{n=0}^{\infty} a_n \cos(n\pi y)\cos(n^2\pi^2 x)$$

where the a_n's are arbitrary constants. We need to choose the constants so that the nonhomogeneous condition ⑦ is satisfied:

$$y^4 - 2y^2 = u(0,y) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi y) \qquad \text{for all } 0 \leq y \leq 1.$$

By problem 4 we should choose $a_0 = \frac{-7}{15}$ and $a_n = \frac{48(-1)^{n+1}}{\pi^4 n^4}$ for $n \geq 1$.

Thus

$$\boxed{u(x,y) = \frac{-7}{15} + \frac{48}{\pi^4}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cos(n\pi y)\cos(n^2\pi^2 x)}{n^4}}$$

solves ①-②-③-④-⑤-⑥-⑦.

<u>Bonus</u>: Let $v = v(x,y)$ be another solution to ①-②-③-④-⑤-⑥-⑦ and consider $w(x,y) = u(x,y) - v(x,y)$. Then w satisfies

$$\begin{cases} w_{xx} + w_{yyyy} \overset{⑧}{=} 0 & \text{if } 0 < x < \infty,\ 0 < y < 1, \\ w_y(x,0) \overset{⑨}{=} 0 \overset{⑩}{=} w_y(x,1) \text{ and } w_{yyy}(x,0) \overset{⑪}{=} 0 \overset{⑫}{=} w_{yyy}(x,1) & \text{if } x \geq 0, \\ w(0,y) \overset{⑬}{=} 0 \overset{⑭}{=} w_x(0,y) & \text{if } 0 \leq y \leq 1. \end{cases}$$

Let $E(x) = \frac{1}{2}\int_0^1 \left[w_x^2(x,y) + w_{yy}^2(x,y)\right]dy$ be the total energy function of the solution w at $x \geq 0$. Then

$$\frac{dE}{dx} = \int_0^1 \left[w_x(x,y)w_{xx}(x,y) + w_{yy}(x,y)w_{yyx}(x,y)\right]dy.$$

Integrating by parts twice gives

$$\int_0^1 w_{yy}(x,y)w_{yyx}(x,y)dy = \left(w_{yy}w_{yx} - w_{yyy}w_x\right)\Big|_{y=0}^{1} + \int_0^1 w_x(x,y)w_{yyyy}(x,y)dy.$$

Differentiating ⑨ and ⑩ with respect to x produces $w_{yx}(x,0) = 0 = w_{yx}(x,1)$ for $x \geq 0$. Using these relations together with ⑪ and ⑫ yields

$$\left(w_{yy}w_{yx} - w_{yyy}w_x\right)\Big|_{y=0}^{1} = w_{yy}(x,1)w_{yx}(x,1) - w_{yyy}(x,1)w_x(x,1) - w_{yy}(x,0)w_{yx}(x,0) + w_{yyy}(x,0)w_x(x,0)$$

$$= 0.$$

Therefore $\frac{dE}{dx} = \int_0^1 w_x(x,y)w_{xx}(x,y)dy + \int_0^1 w_x(x,y)w_{yyyy}(x,y)dy =$

$$\int_0^1 w_x(x,y)\left[w_{xx}(x,y) + w_{yyyy}(x,y)\right]dy = 0$$

7 by ⑧. Consequently $E(x) = E(0)$ for all $x \geq 0$. If we differentiate ⑬ twice with respect to y we obtain $w_{yy}(0,y) \overset{(\dagger)}{=} 0$ for all $0 \leq y \leq 1$. Then

8 $$E(0) = \frac{1}{2}\int_0^1 \left[w_x^2(0,y) + w_{yy}^2(0,y)\right]dy = 0$$

by $(\dagger)$ and (14). Hence, for all $x \geq 0$,

$$0 = E(x) = \frac{1}{2}\int_0^1 \left[w_x^2(x,y) + w_{yy}^2(x,y)\right]dy$$

so the vanishing theorem implies $w_x^2(x,y) + w_{yy}^2(x,y) = 0$ for all $0 \leq y \leq 1$ and all $0 \leq x < \infty$. Thus $w_x(x,y) = 0 = w_{yy}(x,y)$ and it follows that $w(x,y) = c_1 y + c_2$ for some constants c_1 and c_2 and all $0 \leq y \leq 1$, $0 \leq x < \infty$. But ⑬ shows that $c_1 = 0 = c_2$ and so $u(x,y) - v(x,y) = w(x,y) = 0$ for all $0 \leq x < \infty$, $0 \leq y \leq 1$.

That is, $u(x,y) = \dfrac{-7}{15} + \dfrac{48}{\pi^4}\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}\cos(n\pi y)\cos(n^2\pi^2 x)}{n^4}$ is the unique (classical) solution to ①-②-③-④-⑤-⑥-⑦.

#7 Since the PDE, region, and boundary conditions are invariant under rotations (about the origin), we expect a solution that is independent of the spherical coordinate angles θ and φ, and depends only on the radial coordinate r. In this case we would have $\frac{\partial u}{\partial \theta} = 0 = \frac{\partial u}{\partial \varphi}$ and $\frac{\partial^2 u}{\partial \theta^2} = 0 = \frac{\partial^2 u}{\partial \varphi^2}$ and consequently the PDE $\nabla^2 u = 1$ ① would become

4 $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin(\varphi)\cancelto{0}{\frac{\partial u}{\partial \varphi}}\right) + \frac{1}{r^2 \sin^2\varphi}\cancelto{0}{\frac{\partial^2 u}{\partial \theta^2}} = 1$$

or

8 $$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = r^2 .$$

12 Integration gives $r^2 \frac{\partial u}{\partial r} = \frac{r^3}{3} + c_1$ so $\frac{\partial u}{\partial r} = \frac{r}{3} + \frac{c_1}{r^2}$.

16 Integrating again yields $u = \frac{r^2}{6} - \frac{c_1}{r} + c_2$. Applying the B.C. ③ and noting that $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial r}$ for spheres centered at the origin, we have

$$0 = \left.\frac{\partial u}{\partial r}\right|_{r=2} = \left.\left(\frac{r}{3} + \frac{c_1}{r^2}\right)\right|_{r=2} = \frac{2}{3} + \frac{c_1}{4}.$$ Consequently, $c_1 = -\frac{8}{3}$

so $u = \frac{r^2}{6} + \frac{8}{3r} + c_2$. Applying the B.C. ② yields

22 $$0 = u\Big|_{r=1} = \left.\left(\frac{r^2}{6} + \frac{8}{3r} + c_2\right)\right|_{r=1} = \frac{1}{6} + \frac{8}{3} + c_2, \text{ so } c_2 = \frac{-17}{6}.$$

25 Thus $\boxed{u(r,\theta,\varphi) = \frac{r^2}{6} + \frac{8}{3r} - \frac{17}{6}}$ solves ①-②-③.

In cartesian coordinates, the solution is

$$u(x,y,z) = \frac{x^2+y^2+z^2}{6} + \frac{8}{3\sqrt{x^2+y^2+z^2}} - \frac{17}{6}$$

#8 (a) We use separation of variables. We seek nontrivial solutions to ① and the five homogeneous Dirichlet boundary conditions of the form $u(x,y,z) \overset{(*)}{=} X(x)Y(y)Z(z)$. Substituting $(*)$ into ① yields

$$0 = u_{xx} + u_{yy} + u_{zz} = X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z)$$

so $$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = \text{constant} = \lambda \quad \text{and} \quad -\frac{Y''(y)}{Y(y)} = \frac{Z''(z)}{Z(z)} - \lambda =$$

constant $= \mu$. Consequently, we are led to the coupled system

$$X''(x) + \lambda X(x) = 0, \quad X(0) = 0 = X(\pi),$$

$$Y''(y) + \mu Y(y) = 0, \quad Y(0) = 0 = Y(\pi),$$

$$Z''(z) - (\lambda + \mu) Z(z) = 0, \quad Z(0) = 0.$$

(Note: Substituting $(*)$ into the homogeneous Dirichlet B.C. $u(0,y,z) = 0$ for all $0 \le y \le \pi$, $0 \le z \le \pi$, gives $X(0)Y(y)Z(z) = 0$ if $0 \le y \le \pi$, $0 \le z \le \pi$. Consequently $X(0) = 0$ is necessary for a nontrivial solution of the form $(*)$. The other four conditions $X(\pi) = 0 = Y(0) = Y(\pi) = Z(0)$ follow in a similar manner from the homogeneous Dirichlet boundary conditions on the faces $z=0$, $y=0$, $y=\pi$, and $x=\pi$.)

The circled eigenvalue problems in the coupled system above have solutions as follows:

$$\lambda_\ell = \ell^2, \quad X_\ell(x) = \sin(\ell x) \qquad (\ell = 1,2,3,\dots)$$

$$\mu_m = m^2, \quad Y_m(y) = \sin(my) \qquad (m = 1,2,3,\dots).$$

For a given value of λ_ℓ and μ_m, the corresponding solution to

$$Z''_{\ell,m}(z) - (\lambda_\ell + \mu_m) Z_{\ell,m}(z) = 0, \quad Z_{\ell,m}(0) = 0,$$

is (up to a constant factor) $Z_{\ell,m}(z) = \sinh\left(z\sqrt{\ell^2 + m^2}\right)$. Thus

$$u_{\ell,m}(x,y,z) = X_\ell(x)Y_m(y)Z_{\ell,m}(z) = \sin(\ell x)\sin(my)\sinh\left(z\sqrt{\ell^2+m^2}\right)$$

$$(\ell = 1,2,3,\dots \; ; \; m = 1,2,3,\dots)$$

solves ① and the five homogeneous Dirichlet B.C.s, as does the "formal" solution

$$u(x,y,z) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} A_{l,m} \sin(lx) \sin(my) \sinh\left(z\sqrt{l^2+m^2}\right)$$

where each $A_{l,m}$ is a constant. We want to choose the constants so that the nonhomogeneous Dirichlet B.C. ② is met; that is,

$$\sin(x)\left[\frac{3}{4}\sin(y) - \frac{1}{4}\sin(3y)\right] = \sin(x)\sin^3(y) = u(x,y,\pi) \quad \text{for all } 0 \le x \le \pi, \ 0 \le y \le \pi.$$

Hence

19 $$\frac{3}{4}\sin(x)\sin(y) - \frac{1}{4}\sin(x)\sin(3y) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} A_{l,m} \sinh\left(\pi\sqrt{l^2+m^2}\right)\sin(lx)\sin(my)$$

for all $0 \le x \le \pi$, $0 \le y \le \pi$. By inspection,

$$(l=1=m) \qquad \frac{3}{4} = A_{1,1}\sinh(\pi\sqrt{2}) \quad \Rightarrow \quad A_{1,1} = \frac{3}{4\sinh(\pi\sqrt{2})}$$

$$(l=1,\ m=3) \qquad -\frac{1}{4} = A_{1,3}\sinh(\pi\sqrt{10}) \quad \Rightarrow \quad A_{1,3} = \frac{-1}{4\sinh(\pi\sqrt{10})}$$

and all other $A_{l,m} = 0$. This means that

23 $$u(x,y,z) = \frac{3\sin(x)\sin(y)\sinh(z\sqrt{2})}{4\sinh(\pi\sqrt{2})} - \frac{\sin(x)\sin(3y)\sinh(z\sqrt{10})}{4\sinh(\pi\sqrt{10})}$$

solves the problem in part (a).

(b) (Maximum-Minimum Principle for Harmonic Functions) Let D be a bounded open set in $\mathbb{R}^n$ with closure $\bar{D} = D \cup \partial D$. If $u = u(\vec{x})$ is a solution to
24 $\nabla^2 u = 0$ in D such that u is continuous on $\bar{D}$ then

$$\max_{\vec{x}\in\bar{D}} u(\vec{x}) = \max_{\vec{x}\in\partial D} u(\vec{x}) \quad \text{and} \quad \min_{\vec{x}\in\bar{D}} u(\vec{x}) = \min_{\vec{x}\in\partial D} u(\vec{x}).$$

Suppose that $u = v(x,y,z)$ were another solution to the problem in part (a) which is continuous on the cube $0 \le x \le \pi$, $0 \le y \le \pi$, $0 \le z \le \pi$. Then $W(x,y,z) = u(x,y,z) - v(x,y,z)$ is a solution to

$$\nabla^2 W = 0 \quad \text{in} \quad 0 < x < \pi,\ 0 < y < \pi,\ 0 < z < \pi,$$

$$W(x,y,z) = 0 \quad \text{on the six boundary faces of the cube,}$$

and W is continuous on $0 \le x \le \pi$, $0 \le y \le \pi$, $0 \le z \le \pi$. By the maximum-minimum principle for harmonic functions, $W = 0$ on $0 \le x \le \pi$, $0 \le y \le \pi$, $0 \le z \le \pi$. Therefore $v(x,y,z) = u(x,y,z) = \dfrac{3\sin(x)\sin(y)\sinh(z\sqrt{2})}{4\sinh(\pi\sqrt{2})} - \dfrac{\sin(x)\sin(3y)\sinh(z\sqrt{10})}{4\sinh(\pi\sqrt{10})}$

so there is only one solution to the problem in part (a).

25

Math 325

Final Exam

Summer 2011

$\mu = 137.1$

$\sigma = 43.1$

$n = 23$

Distribution of Scores

Range	Graduate Letter Grade	Undergraduate Letter Grade	Frequency
174 – 200	A	A	6
146 – 173	B	B	5
120 – 145	C	B	5
100 – 119	C	C	2
0 – 99	F	D	5