Mathematics 325 | Final Exam Summer 2012 | Name: Dr. Grow

1.(28 pts.) (a) Verify that $u_p(x,y) = e^{xy}$ is a particular solution to the linear, nonhomogeneous partial differential equation $yu_x - xu_y = (y^2 - x^2)e^{xy}$.

(b) Find the general solution of the linear, homogeneous partial differential equation $yu_x - xu_y = 0$.

(c) Find the solution of the partial differential equation $yu_x - xu_y = (y^2 - x^2)e^{xy}$ satisfying the auxiliary condition $u(x,0) = x^6 + 1$ for all real x.

2.(28 pts.) Classify the type - elliptic, parabolic, or hyperbolic - of the partial differential equation $u_{xx} + u_{yy} - 2u_{xy} = \sin(x - y)$ and find, if possible, the general solution in the xy-plane.

3.(29 pts.) A homogeneous solid material occupying $D = \{(x,y,z) \in \mathbb{R}^3 : 4 \le x^2 + y^2 + z^2 \le 100\}$ is completely insulated and its initial temperature at position (x,y,z) in D is $200/\sqrt{x^2+y^2+z^2}$.

(a) Write, without proof or derivation, the partial differential equation and initial/boundary conditions that completely govern the temperature $u(x,y,z,t)$ at position (x,y,z) in D and time $t \ge 0$.

(b) Use the divergence theorem to help show that the heat energy $H(t) = \iiint_D c\rho u(x,y,z,t)\,dV$ of the material in D at time t is a constant function of time. Here c and ρ denote the (positive, constant) specific heat and mass density, respectively, of the material in D.

(c) Compute the (constant) steady-state temperature that the material in D reaches after a long time.

4.(28 pts.) Solve $u_t - u_{xx} = 0$ in the upper half-plane $-\infty < x < \infty,\ 0 < t < \infty$, subject to the initial condition $u(x,0) = x^2$ if $-\infty < x < \infty$. Note: You may find useful the identity $\int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp = \sqrt{\pi}/2$.

5.(29 pts.) Let $f(x,t) = \begin{cases} \sqrt{\pi/2} & \text{if } |x| < |t|, \\ 0 & \text{otherwise.} \end{cases}$ Use Fourier transform methods to solve

$u_{tt} - u_{xx} \overset{1}{=} f(x,t)$ in the xt-plane subject to the initial conditions $u(x,0) \overset{2}{=} 0 \overset{3}{=} u_t(x,0)$ for all real x.

6.(29 pts.) (a) Show that the 2π-periodic function determined by the formula $f(x) = x^2$ for x in the interval $[-\pi,\pi]$ has full Fourier series $\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(nx)}{n^2}$.

(b) Discuss the pointwise convergence or lack thereof for the full Fourier series of f at each point x in the interval $[-\pi,\pi]$.

(c) Use the results of parts (a) and (b) to help find the sums of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

7.(29 pts.) (a) Find a solution to the damped wave equation

(1) $$u_{tt} - u_{xx} + 2u_t = 0 \text{ in } 0 < x < \pi,\ 0 < t < \infty,$$

satisfying the boundary conditions

(2)-(3) $$u_x(0,t) = 0 = u_x(\pi,t) \text{ for } 0 \le t < \infty,$$

(4) $$u(x,0) = 0 \text{ for } 0 \le x \le \pi,$$

(5) $$u_t(x,0) = x^2 \text{ for } 0 \le x \le \pi.$$

(b) Show that if $u = u(x,t)$ satisfies (1)-(2)-(3) then its energy function

$$E(t) = \frac{1}{2}\int_0^{\pi}\left[u_t^2(x,t) + u_x^2(x,t)\right]dx$$

is decreasing on $0 \le t < \infty$.

(c) Is the solution to the problem in part (a) unique? Justify your answer.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$
A.	$\begin{cases} 1 & \text{if } -b<x<b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c<x<d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi}-e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2+a^2} \quad (a>0)$	$\sqrt{\frac{\pi}{2}}\frac{e^{-a\lvert\xi\rvert}}{a}$
D.	$\begin{cases} x & \text{if } 0<x\le b, \\ 2b-x & \text{if } b<x<2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1+2e^{-ib\xi}-e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x>0, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{1}{(a+i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b<x<c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\xi)c}-e^{(a-i\xi)b}}{(a-i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b<x<b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(b(\xi-a))}{\xi-a}$
H.	$\begin{cases} e^{iax} & \text{if } c<x<d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{ic(a-\xi)}-e^{id(a-\xi)}}{i(\xi-a)\sqrt{2\pi}}$
I.	$e^{-ax^2} \quad (a>0)$	$\frac{1}{\sqrt{2a}}e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x} \quad (a>0)$	$\begin{cases} 0 & \text{if } \lvert\xi\rvert\ge a, \\ \sqrt{\frac{\pi}{2}} & \text{if } \lvert\xi\rvert<a. \end{cases}$

Fourier Series Convergence Theorems

Consider the eigenvalue problem

(1) $$X''(x)+\lambda X(x)=0 \text{ in } a<x<b$$

with any symmetric boundary conditions of the form

(2) $$\begin{cases}\alpha_1 f(a)+\beta_1 f(b)+\gamma_1 f'(a)+\delta_1 f'(b)=0\\ \alpha_2 f(a)+\beta_2 f(b)+\gamma_2 f'(a)+\delta_2 f'(b)=0\end{cases}$$

and let $\Phi=\{X_1,X_2,X_3,\ldots\}$ be the complete orthogonal set of eigenfunctions for (1)-(2). Let f be any absolutely integrable function defined on $a\le x\le b$. Consider the Fourier series for f with respect to Φ:

$$\sum_{n=1}^{\infty} A_n X_n(x)$$

where

$$A_n=\frac{\langle f,X_n\rangle}{\langle X_n,X_n\rangle}\qquad (n=1,2,3,\ldots).$$

Theorem 2. (Uniform Convergence) If

(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a\le x\le b$ and

(ii) f satisfies the given symmetric boundary conditions,

then the Fourier series of f converges uniformly to f on $[a,b]$.

Theorem 3. (L^2 −Convergence) If

$$\int_a^b |f(x)|^2\,dx<\infty$$

then the Fourier series of f converges to f in the mean-square sense in (a,b).

Theorem 4. (Pointwise Convergence of Classical Fourier Series)

(i) If f is a continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) at x converges pointwise to $f(x)$ in the open interval $a<x<b$.

(ii) If f is a piecewise continuous function on $a\le x\le b$ and f' is piecewise continuous on $a\le x\le b$, then the classical Fourier series (full, sine, or cosine) converges pointwise at every point x in $(-\infty,\infty)$. The sum of the Fourier series is

$$\sum_{n=1}^{\infty} A_n X_n(x)=\frac{f(x^+)+f(x^-)}{2}$$

for all x in the open interval (a,b).

Theorem 4∞. If f is a function of period $2l$ on the real line for which f and f' are piecewise continuous, then the classical full Fourier series converges to $\dfrac{f(x^+)+f(x^-)}{2}$ for every real x.

#1. (a) $u_p = e^{xy}$ so $(u_p)_x = y e^{xy}$ and $(u_p)_y = x e^{xy}$. Therefore

$$y(u_p)_x - x(u_p)_y = y^2 e^{xy} - x^2 e^{xy} = (y^2 - x^2)e^{xy}.$$

(b) Along characteristic curves $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$, solutions to $a(x,y)u_x + b(x,y)u_y = 0$ are constant. Therefore the characteristic curves of $y u_x - x u_y = 0$ are given by $\frac{dy}{dx} = \frac{-x}{y}$. Separating variables and integrating yields

$$\frac{y^2}{2} + c_1 = \int y\,dy = -\int x\,dx = -\frac{x^2}{2} + c_2 .$$

Rearranging, $y^2 + x^2 = c$ where c is an arbitrary constant. When $c > 0$, the characteristic curves are circles centered at the origin. Along such a circle, solutions $u = u(x,y)$ are constant:

$$u(x, y(x)) = u(x, \pm\sqrt{c - x^2}) = u(0, \pm\sqrt{c}) = f(c).$$

The general solution of $yu_x - xu_y = 0$ is

$$\boxed{u(x,y) = f(x^2 + y^2)}$$

where f is a general C^1-function of a single real variable.

(c) The general solution of $yu_x - xu_y = (y^2 - x^2)e^{xy}$ is $u = u_c + u_p$ where u_c is the general solution of the associated homogeneous equation $yu_x - xu_y = 0$ and u_p is any particular solution of the nonhomogeneous equation $yu_x - xu_y = (y^2 - x^2)e^{xy}$. Thus $u(x,y) = f(x^2 + y^2) + e^{xy}$ is the general solution of $yu_x - xu_y = (y^2 - x^2)e^{xy}$. To satisfy the auxiliary condition, f must be chosen such that

$$x^6 + 1 = u(x,0) = f(x^2 + 0^2) + e^{x\cdot 0} = f(x^2) + 1$$

for all real x. Therefore $f(w) = w^3$ for all $w \geq 0$ so

$$\boxed{u(x,y) = (x^2 + y^2)^3 + e^{xy}}.$$

#2. $u_{xx} - 2u_{xy} + u_{yy} = \sin(x-y)$

2 $B^2 - 4AC = (-2)^2 - 4(1)(1) = 0$. Therefore the equation is of $\boxed{\text{parabolic}}$ type.

The pde can be expressed in the form

$$\left(\frac{\partial^2}{\partial x^2} - 2\frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2}\right)u = \sin(x-y)$$

or

3 $$(*) \qquad \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u = \sin(x-y).$$

In order to solve the pde we make the change-of-coordinates

6 $$\begin{cases} \xi = -(\beta x - \alpha y) = -(-x-y) = x+y, \\ \eta = \alpha x + \beta y = x - y. \end{cases}$$

By the chain rule, for any C^1-function v we have

2 $$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta},$$
$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial v}{\partial \xi} - \frac{\partial v}{\partial \eta}.$$

Consequently, as operators

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta},$$

so $\frac{\partial}{\partial x} - \frac{\partial}{\partial y} = 2\frac{\partial}{\partial \eta}$. Substituting in (*) yields

3 $$4\frac{\partial^2 u}{\partial \eta^2} = \left(2\frac{\partial}{\partial \eta}\right)\left(2\frac{\partial}{\partial \eta}\right)u = \sin(\eta).$$

Integrating twice produces

(Integrate w.r.t. η holding ξ fixed.)

2 $$\frac{\partial u}{\partial \eta} = \int \frac{1}{4}\sin(\eta)\,d\eta = -\frac{1}{4}\cos(\eta) + c_1(\xi)$$

#2 (Cont.) and $u = \int\left[-\frac{1}{4}\cos(\eta) + c_1(\xi)\right]d\eta = -\frac{1}{4}\sin(\eta) + \eta c_1(\xi) + c_2(\xi)$.

Using the change-of-coordinates equation yields

$$\boxed{u(x,y) = -\frac{1}{4}\sin(x-y) + (x-y)f(x+y) + g(x+y)}$$

where f and g are C^2-functions of a single real variable.

#3. (a) The pde and auxiliary conditions governing the temperature $u(x,y,z,t)$ at position (x,y,z) in D at time $t \geq 0$ are:

3-D heat equation

$$\frac{\partial u}{\partial t} - k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0 \quad \text{if } 4 < x^2+y^2+z^2 < 100 \text{ and } t > 0,$$

10

heat flux across boundary

$$\nabla u \cdot \vec{n} = 0 \quad \text{if } x^2+y^2+z^2 = 4 \text{ or } x^2+y^2+z^2 = 100 \text{ and } t \geq 0,$$

initial temperature distribution

$$u(x,y,z,0) = \frac{200}{\sqrt{x^2+y^2+z^2}} \quad \text{if } 4 \leq x^2+y^2+z^2 \leq 100.$$

6 pts. → (b) Let $H(t) = \iiint_D c\rho u(x,y,z,t)\,dV$ be the heat energy of the material in D at time $t \geq 0$ where c and ρ are the (positive, constant) specific heat and mass density of the material in D. Then

$$\frac{dH}{dt} = \frac{d}{dt}\iiint_D c\rho u(x,y,z,t)\,dV = \iiint_D c\rho \frac{\partial u}{\partial t}(x,y,z,t)\,dV =$$

$$\iiint_D c\rho k \nabla\cdot(\nabla u)\,dV = \iint_{\partial D} c\rho k \nabla u \cdot \vec{n}\,dS = \iint_{\partial D} 0\,dS = 0.$$

Therefore $H(t) = H(0)$ for all $t \geq 0$.

#3(c) Assume $U = \lim_{t\to\infty} u(x,y,z,t)$ is the (constant) steady-state temperature reached by the material in D at position (x,y,z). Then

$$H(0) = \lim_{t\to\infty} H(t) = \lim_{t\to\infty} \iiint_D c\rho u(x,y,z,t)\,dV = \iiint_D c\rho \lim_{t\to\infty} u(x,y,z,t)\,dV$$

$$= \iiint_D c\rho U\,dV = c\rho U \,\text{volume}(D). \quad \text{But}$$

$$\text{volume}(D) = \iiint_D dV = \int_0^{2\pi}\int_0^{\pi}\int_2^{10} r^2 \sin\varphi\, dr\, d\varphi\, d\theta = \left(\int_0^{\pi} \sin\varphi\, d\varphi\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_2^{10} r^2\, dr\right)$$

$$= (-\cos\varphi)\Big|_0^{\pi} (2\pi)\left(\frac{r^3}{3}\Big|_2^{10}\right) = \frac{4\pi}{3}(1000-8) = \frac{4\pi}{3}(992).$$

and

$$H(0) = \iiint_D c\rho u(x,y,z,0)\,dV = \iiint_D c\rho \frac{200}{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_2^{10} c\rho \frac{200}{r} \cdot r^2 \sin\varphi\, dr\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} c\rho\, 100 r^2 \Big|_2^{10} \sin\varphi\, d\varphi\, d\theta = 2\pi c\rho (100)(96)(-\cos\varphi)\Big|_0^{\pi} = 4\pi c\rho(9600).$$

Therefore $$U = \frac{H(0)}{c\rho\, \text{volume}(D)} = \frac{4\pi c\rho(9600)}{\frac{4\pi}{3} c\rho(992)} = \frac{28800}{992} = \boxed{\frac{900}{31}} \cong 29.$$

#4. A solution of $u_t - k u_{xx} = 0$ in $-\infty < x < \infty$, $0 < t < \infty$, satisfying $u(x,0) = \varphi(x)$ for all real x is

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} \varphi(y) e^{-\frac{(x-y)^2}{4kt}}\,dy \qquad (t > 0).$$

Taking $k=1$ and $\varphi(y) = y^2$ for our problem, we find

#4 (cont.) $u(x,t) = \dfrac{1}{\sqrt{4\pi t}} \displaystyle\int_{-\infty}^{\infty} y^2 e^{-\frac{(x-y)^2}{4t}}\, dy$. Make the change of variables $p = \dfrac{y-x}{\sqrt{4t}}$. Then $dp = \dfrac{dy}{\sqrt{4t}}$, $y \to +\infty$ implies $p \to +\infty$, and $y \to -\infty$ implies $p \to -\infty$. Therefore

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x + p\sqrt{4t})^2 e^{-p^2} dp = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x^2 + 2px\sqrt{4t} + 4p^2 t) e^{-p^2} dp$$

$$= \frac{x^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp + \frac{2x\sqrt{4t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} p e^{-p^2} dp + \frac{4t}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2} dp .$$

But $\displaystyle\int_{-\infty}^{\infty} e^{-p^2} dp = \sqrt{\pi}$, $\displaystyle\int_{-\infty}^{\infty} p e^{-p^2} dp = 0$ since $f(p) = pe^{-p^2}$ is an odd function and

$$\int_{-\infty}^{\infty} p^2 e^{-p^2} dp = \int_{-\infty}^{\infty} \underbrace{p}_{v} \cdot \underbrace{p e^{-p^2} dp}_{dv} = \cancel{p\left(-\tfrac{1}{2} e^{-p^2}\right)\Big|_{-\infty}^{\infty}}^{\,0} - \int_{-\infty}^{\infty} -\tfrac{1}{2} e^{-p^2} dp = \frac{\sqrt{\pi}}{2} .$$

Consequently $\boxed{u(x,t) = x^2 + 2t}$. Since $\varphi(x) = x^2$ is not bounded on $-\infty < x < \infty$, we need to check our answer since technically the formula for $u = u(x,t)$ is not guaranteed to solve the initial value problem in that case. But $u(x,t) = x^2 + 2t$ implies $u_t = 2$ and $u_{xx} = 2$ so $u_t - u_{xx} = 2 - 2 = 0$ for all points (x,t). Also $u(x,t) = x^2 + 2t$ implies $u(x,0) = x^2$ for all real x. Therefore the boxed function u above solves the initial value problem.

#5. (Method 1) Let $u=u(x,t)$ solve ①-②-③. We take the Fourier transform of both sides of ① with respect to x (holding $t>0$ fixed):

$$\frac{\partial^2}{\partial t^2}\mathcal{F}(u)(\xi)-(i\xi)^2\mathcal{F}(u)(\xi)=\mathcal{F}(u_{tt}-u_{xx})(\xi)=\mathcal{F}f(\xi)=\hat{f}(\xi,t)$$

$$(*)\qquad \frac{\partial^2}{\partial t^2}\mathcal{F}(u)(\xi)+\xi^2\mathcal{F}(u)(\xi)=\hat{f}(\xi,t).$$

The general solution of the second-order, linear, nonhomogeneous ODE $(*)$ in t (with parameter ξ) is

$$\mathcal{F}(u)(\xi)=U_h(\xi,t)+U_p(\xi,t)$$

where $U_h(\xi,t)=c_1(\xi)\cos(\xi t)+c_2(\xi)\sin(\xi t)$ is the general solution of the associated homogeneous equation of $(*)$, $\frac{\partial^2}{\partial t^2}\mathcal{F}(u)(\xi)+\xi^2\mathcal{F}(u)(\xi)\overset{(**)}{=}0$, and $U_p(\xi,t)$ is a particular solution of $(*)$. We use variation of parameters to write

$$U_p(\xi,t)=v_1(\xi,t)\cos(\xi t)+v_2(\xi,t)\sin(\xi t)$$

where

$$v_1(\xi,t)=\int_0^t\frac{-Fy_2}{W}\,d\tau=\int_0^t-\frac{\hat{f}(\xi,\tau)\sin(\xi\tau)}{W(\xi,\tau)}\,d\tau,$$

$$v_2(\xi,t)=\int_0^t\frac{Fy_1}{W}\,d\tau=\int_0^t\frac{\hat{f}(\xi,\tau)\cos(\xi\tau)}{W(\xi,\tau)}\,d\tau,$$

$y_1=\cos(\xi t)$, $y_2=\sin(\xi t)$ form a fundamental set of solutions of $(**)$, $F=\hat{f}(\xi,t)$, and

$$W=W(\xi,t)=\begin{vmatrix}\cos(\xi t) & \sin(\xi t)\\ \frac{\partial}{\partial t}\cos(\xi t) & \frac{\partial}{\partial t}\sin(\xi t)\end{vmatrix}=\begin{vmatrix}\cos(\xi t) & \sin(\xi t)\\ -\xi\sin(\xi t) & \xi\cos(\xi t)\end{vmatrix}=\xi$$

is the Wronskian of y_1 and y_2. Thus

$$\mathcal{F}(u)(\xi)=c_1(\xi)\cos(\xi t)+c_2(\xi)\sin(\xi t)-\cos(\xi t)\int_0^t\frac{\hat{f}(\xi,\tau)\sin(\xi\tau)}{\xi}\,d\tau+\sin(\xi t)\int_0^t\frac{\hat{f}(\xi,\tau)\cos(\xi\tau)}{\xi}\,d\tau$$

$$=c_1(\xi)\cos(\xi t)+c_2(\xi)\sin(\xi t)+\frac{1}{\xi}\int_0^t\hat{f}(\xi,\tau)\sin(\xi t-\tau)\,d\tau.$$

(15 pts to here)

#5 (Method 1, cont.) Applying the auxiliary condition (2) yields

(1) $$0 = \mathcal{F}(u)(\xi)\Big|_{t=0} = c_1(\xi) + 0 + 0 = c_1(\xi).$$

Next, observe that

$$\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) = -\xi c_1(\xi)\sin(\xi t) + \xi c_2(\xi)\cos(\xi t) - \xi\sin(\xi t)\int_0^t \frac{\hat{f}(\xi,\tau)\sin(\xi\tau)}{\xi}\,d\tau$$
$$- \cos(\xi t)\cdot\frac{\hat{f}(\xi,t)\sin(\xi t)}{\xi} + \xi\cos(\xi t)\int_0^t \frac{\hat{f}(\xi,\tau)\cos(\xi\tau)}{\xi}\,d\tau$$
$$+ \sin(\xi t)\cdot\frac{\hat{f}(\xi,t)\cos(\xi t)}{\xi}$$

so by (3) we have

(1) $$0 = \mathcal{F}(u_t)(\xi)\Big|_{t=0} = \frac{\partial\mathcal{F}}{\partial t}(u)(\xi)\Big|_{t=0} = 0 + \xi c_2(\xi) - 0 - 0 + 0 + 0 = \xi c_2(\xi).$$

(2) Thus $$\mathcal{F}(u)(\xi) = \frac{1}{\xi}\int_0^t \hat{f}(\xi,\tau)\sin(\xi(t-\tau))\,d\tau. \qquad (\dagger)$$

Using formula A in the table of Fourier transforms with $b = t-\tau$, we see that

(1) $$\frac{\sin(\xi(t-\tau))}{\xi} = \sqrt{\frac{\pi}{2}}\,\mathcal{F}\left(\chi_{(-(t-\tau),\,t-\tau)}\right)(\xi).$$

Also, since $f(x,t) = \sqrt{\frac{\pi}{2}}\,\chi_{(-t,t)}(x)$,

(1) $$\hat{f}(\xi,\tau) = \sqrt{\frac{\pi}{2}}\,\mathcal{F}\left(\chi_{(-\tau,\tau)}\right)(\xi).$$

Substituting in $(\dagger)$ and using the convolution formula $\mathcal{F}(g*h)(\xi) = \sqrt{2\pi}\,\mathcal{F}(g)(\xi)\mathcal{F}(h)(\xi)$ leads to

$$\mathcal{F}(u)(\xi) = \int_0^t \sqrt{\frac{\pi}{2}}\,\mathcal{F}\left(\chi_{(-\tau,\tau)}\right)(\xi)\sqrt{\frac{\pi}{2}}\,\mathcal{F}\left(\chi_{(-(t-\tau),\,t-\tau)}\right)(\xi)\,d\tau$$

(3) $$= \frac{\pi}{2}\int_0^t \frac{1}{\sqrt{2\pi}}\mathcal{F}\left(\chi_{(-\tau,\tau)} * \chi_{(-(t-\tau),\,t-\tau)}\right)(\xi)\,d\tau = \mathcal{F}\left(\frac{\sqrt{\pi}}{2\sqrt{2}}\int_0^t \chi_{(-\tau,\tau)} * \chi_{(-(t-\tau),\,t-\tau)}\,d\tau\right)(\xi).$$

(23 pts. to here)

#5. (Method 1, cont.) The inversion theorem then implies

$$u(x,t) = \frac{\sqrt{\pi}}{2\sqrt{2}} \int_0^t \left(\chi_{(-\tau,\tau)} * \chi_{(-(t-\tau),\, t-\tau)} \right)(x)\, d\tau .$$

$$\therefore \quad u(x,t) = \frac{\sqrt{\pi}}{2\sqrt{2}} \int_0^t \int_{-\infty}^{\infty} \chi_{(-\tau,\tau)}(s)\, \chi_{(-(t-\tau),\, t-\tau)}(x-s)\, ds\, d\tau$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}} \int_0^t \int_{-\infty}^{\infty} \chi_{(-\tau,\tau)}(s)\, \chi_{(x-(t-\tau),\, x+t-\tau)}(s)\, ds\, d\tau$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}} \left(\int_0^t \text{length of } \left\{ (-\tau,\tau) \cap (x-(t-\tau),\, x+t-\tau) \right\} d\tau \right) \chi_{(-t,t)}(x)$$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}} \left[\int_0^{\frac{t-x}{2}} 2\tau\, d\tau + \int_{\frac{t-x}{2}}^{\frac{t+x}{2}} (t-x)\, d\tau + \int_{\frac{t+x}{2}}^{t} 2(t-\tau)\, d\tau \right] \chi_{(-t,t)}(x).$$

$$\therefore \quad \boxed{u(x,t) = \frac{\sqrt{\pi}}{4\sqrt{2}} (t^2 - x^2)\, \chi_{(-t,t)}(x)} \quad \text{if } -\infty < x < \infty \text{ and } 0 < t < \infty .$$

Since $f(x,-t) = f(x,t)$ and the pde $u_{tt} - u_{xx} = f(x,t)$ is invariant under reflection through the x-axis, $(x,t) \mapsto (x,-t)$, it follows that

$$\boxed{u(x,t) = \frac{\sqrt{\pi}}{4\sqrt{2}} (t^2 - x^2)\, \chi_{(-|t|,|t|)}(x)}$$

(27 pts. to here)

solves ①-②-③ in the entire xt-plane.

Note: This is a generalized solution to ①-②-③ in the xt-plane. In particular, $u(x,t) = \frac{\sqrt{\pi}}{4\sqrt{2}} (t^2 - x^2)\chi_{(-|t|,|t|)}(x)$ is a continuous function in the xt-plane such that

$$u_{tt}(x,t) - u_{xx}(x,t) = \left\{ \begin{array}{ll} \sqrt{\pi/2} & \text{if } |x| < |t| \\ 0 & \text{if } |x| > |t| \end{array} \right\} = f(x,t) \quad \text{if } |x| \neq |t|$$

and $u(x,0) = 0 = u_t(x,0)$ if $-\infty < x < \infty$.

#5. (Method 2) Let $u = u(x,t)$ solve ①-②-③ in the upper half-plane $-\infty < x < \infty$ and $t > 0$. Proceeding as in the method 1 solution, we find that the Fourier transform of u with respect to x (holding $t > 0$ fixed) is

$$(\dagger) \qquad \mathcal{F}(u)(\xi) = \frac{1}{\xi}\int_0^t \hat{f}(\xi,\tau)\sin(\xi(t-\tau))\,d\tau .$$

But $f(x,t) = \sqrt{\frac{\pi}{2}}\,\chi_{(-t,t)}(x)$, so formula A in the Fourier transform table implies $\hat{f}(\xi,\tau) = \dfrac{\sin(\tau\xi)}{\xi}$. Substituting in $(\dagger)$ gives

$$\mathcal{F}(u)(\xi) = \frac{1}{\xi^2}\int_0^t \sin(\tau\xi)\sin(\xi(t-\tau))\,d\tau .$$

Using the identity $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$ gives

$$\mathcal{F}(u)(\xi) = \frac{1}{2\xi^2}\int_0^t \left[\cos(2\xi\tau - \xi t) - \cos(\xi t)\right]d\tau$$

$$= \frac{1}{2\xi^2}\left[\frac{\sin(2\xi\tau - \xi t)}{2\xi} - \tau\cos(\xi t)\right]\Bigg|_{\tau=0}^{t}$$

$$= \frac{1}{2\xi^2}\left[\frac{\sin(\xi t)}{\xi} - t\cos(\xi t)\right].$$

Taking the inverse transform yields

$$u(x,t) = \lim_{M\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-M}^{M}\mathcal{F}(u)(\xi)e^{ix\xi}\,d\xi$$

$$= \lim_{M\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-M}^{M}\overbrace{\frac{1}{2\xi^2}\left[\frac{\sin(\xi t)}{\xi} - t\cos(\xi t)\right]}^{\text{even function of }\xi}\overbrace{e^{ix\xi}}^{\cos(x\xi)+i\sin(x\xi)}\,d\xi$$

$$= \lim_{M\to\infty}\frac{2}{\sqrt{2\pi}}\int_0^M\left(\frac{\sin(\xi t)}{2\xi^3} - \frac{t\cos(\xi t)}{2\xi^2}\right)\cos(\xi x)\,d\xi$$

Using the identities $\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$ and $\cos A\cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$ yields

23 pts. to here

#5 (Method 2)(cont.)

$$u(x,t) = \lim_{M\to\infty} \frac{2}{\sqrt{2\pi}} \int_0^M \left[\frac{1}{4\xi^3}\Big(\sin(\xi(t+x)) + \sin(\xi(t-x))\Big) - \frac{\xi t}{4\xi^3}\Big(\cos(\xi(t+x)) + \cos(\xi(t-x))\Big)\right] d\xi$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{\sin(\xi(t-x)) - \xi(t-x)\cos(\xi(t-x))}{4\xi^3}\, d\xi + \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{\sin(\xi(t+x)) - \xi(t+x)\cos(\xi(t+x))}{4\xi^3}\, d\xi$$

$$+ \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{\xi x\cos(\xi(t+x)) - \xi x\cos(\xi(t-x))}{4\xi^3}\, d\xi$$

$$= \frac{(t-x)^2}{2\sqrt{2\pi}} \int_0^\infty \frac{\sin(\xi(t-x)) - \xi(t-x)\cos(\xi(t-x))}{[\xi(t-x)]^3}\, d(\xi(t-x)) + \frac{(t+x)^2}{2\sqrt{2\pi}} \int_0^\infty \frac{\sin(\xi(t+x)) - \xi(t+x)\cos(\xi(t+x))}{[\xi(t+x)]^3}\, d(\xi(t+x))$$

$$+ \frac{x}{2\sqrt{2\pi}} \int_0^\infty \frac{\cos(\xi(t+x)) - \cos(\xi(t-x))}{\xi^2}\, d\xi .$$

Using the identities $\displaystyle\int_0^\infty \frac{\sin(\lambda) - \lambda\cos(\lambda)}{\lambda^3}\, d\lambda = \frac{\pi}{4}$ (see Gradshteyn & Ryzhik 3.784 (7))

and $\displaystyle\int_0^\infty \frac{\cos(a\xi) - \cos(b\xi)}{\xi^2}\, d\xi = \frac{(b-a)\pi}{2}$ (see Gradshteyn & Ryzhik 3.784(3,

leads to

$$u(x,t) = \frac{(t-x)^2}{2\sqrt{2\pi}} \cdot \frac{\pi}{4} + \frac{(t+x)^2}{2\sqrt{2\pi}} \cdot \frac{\pi}{4} + \frac{x}{2\sqrt{2\pi}} \cdot \frac{(-2x)\pi}{2}$$

$$= \frac{\sqrt{\pi}}{8\sqrt{2}} \left[t^2 - \cancel{2xt} + x^2 + t^2 + \cancel{2xt} + x^2 - 4x^2 \right]$$

$$= \frac{\sqrt{\pi}}{4\sqrt{2}} (t^2 - x^2).$$ (Valid if $|x| < t$ and $t > 0$.)

Proceeding as in method 1, it follows that $\boxed{u(x,t) = \frac{\sqrt{\pi}}{4\sqrt{2}}(t^2 - x^2)\chi_{(-|t|,|t|)}(x)}$

in the xt-plane.

#6. (a) Since f is an even function, the full Fourier series on $(-\pi,\pi)$:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right]$$

is actually a cosine series since

$$b_n = \frac{\langle f, \sin(n\cdot)\rangle}{\langle \sin(n\cdot), \sin(n\cdot)\rangle} = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overbrace{f(x)}^{\text{even}}\overbrace{\sin(nx)}^{\text{odd}}}_{\text{odd}}\,dx = 0 \qquad (n=1,2,3,\dots).$$

We also have

$$a_0 = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{x^3}{3\pi}\Big|_0^{\pi} = \frac{\pi^2}{3}$$

and if $n \geq 1$,

$$a_n = \frac{\langle f, \cos(n\cdot)\rangle}{\langle \cos(n\cdot), \cos(n\cdot)\rangle} = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overbrace{f(x)}^{\text{even}}\overbrace{\cos(nx)}^{\text{even}}}_{\text{even}}\,dx = \frac{2}{\pi}\int_0^{\pi} \underbrace{x^2}_{u}\underbrace{\cos(nx)\,dx}_{dv} = \frac{2}{\pi}\left[\cancel{\frac{x^2\sin(nx)}{n}\Big|_0^{\pi}}^{\,0} - \frac{2}{n}\int_0^{\pi} \underbrace{x}_{u}\underbrace{\sin(nx)\,dx}_{dv}\right]$$

$$= \frac{-4}{\pi n}\left[-\frac{x\cos(nx)}{n}\Big|_0^{\pi} - \cancel{\int_0^{\pi} -\frac{\cos(nx)}{n}\,dx}^{\,0}\right] = \frac{4\pi\cos(\pi n)}{\pi n^2} = \frac{4(-1)^n}{n^2}.$$ Therefore the full

Fourier series of f is

$$\boxed{\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n\cos(nx)}{n^2}.}$$

(b) It is clear that f is continuous and f' is piecewise continuous on the real line. Since f is 2π-periodic as well, Theorem 4^{∞} from the Fourier series Convergence theorems guarantees that the full Fourier series of f converges to $\frac{f(x^+)+f(x^-)}{2}$ for every real x. But $f(x^+)=f(x)$ and $f(x^-)=f(x)$ for every real x since f is continuous on the real line. Therefore the full Fourier series of f converges to $f(x)$ for all real x. In particular,

$$\boxed{x^2 = f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)}$$

for all $-\pi \leq x \leq \pi$.

#6 (c) Taking $x=0$ in the identity from part (b) gives

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \quad \text{so} \quad \boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}}.$$

Taking $x=\pi$ in the identity from part (b) leads to

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \overbrace{\cos(n\pi)}^{(-1)^n}}{n^2} \quad \text{so} \quad \boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4}\left(\pi^2 - \frac{\pi^2}{3}\right) = \frac{\pi^2}{6}}.$$

#7 (a) We seek nontrivial solutions of the form $u(x,t) = X(x)T(t)$ to (1)-(2)-(3)-(4). Substituting in (1) yields $X(x)T''(t) - X''(x)T(t) + 2X(x)T'(t) = 0$ so

$$(*) \qquad \frac{-X''(x)}{X(x)} = \frac{-\left(T''(t) + 2T'(t)\right)}{T(t)} = \text{constant} = \lambda.$$

Substituting $u(x,t) = X(x)T(t)$ in (2) and (3) gives $X'(0)T(t) = 0 = X'(\pi)T(t)$ for $t \geq 0$ and in (4) gives $X(x)T(0) = 0$ for $0 \leq x \leq \pi$. Thus nontriviality of u and $(*)$ imply

$$(6) \qquad X''(x) + \lambda X(x) = 0, \quad X'(0) = 0 = X'(\pi),$$

$$(7) \qquad T''(t) + 2T'(t) + \lambda T(t) = 0, \quad T(0) = 0.$$

By Sec. 4.2, the eigenvalues and eigenfunctions of (6) are $\lambda_n = n^2$ and $X_n(x) = \cos(nx)$ where $n = 0, 1, 2, \ldots$ Substituting $\lambda = \lambda_n = n^2$ in (6) yields

$$(8) \qquad T_n''(t) + 2T_n'(t) + n^2 T_n(t) = 0, \quad T_n(0) = 0.$$

We look for solutions to the differential equation in (8) of the form $T_n(t) = e^{r_n t}$ where r_n is a constant. Substituting $T_n(t) = e^{r_n t}$ in (8) leads to

$$r_n^2 + 2r_n + n^2 = 0 \quad \text{so} \quad r_n = -1 \pm \sqrt{1-n^2}.$$

If $n=0$ then $r_0 = 0$ or $r_0 = -2$ so $T_0(t) = c_1 + c_2 e^{-2t}$ is the general solution of the DE in (8).

If $n=1$ then $r_1 = -1$ with multiplicity two so $T_1(t) = c_1 e^{-t} + c_2 t e^{-t}$ is the general solution of the DE in (8).

#7 (a) (cont.)

If $n \geq 2$ then $r_n = -1 \pm i\sqrt{n^2-1}$ so $T_n(t) = e^{-t}\left(c_1\cos(\sqrt{n^2-1}\,t) + c_2\sin(\sqrt{n^2-1}\,t)\right)$ is the general solution of the DE in (8). Taking into account the initial condition in (8), we have, up to a constant factor,

$$T_0(t) = 1 - e^{-2t}, \quad T_1(t) = te^{-t}, \quad \text{and} \quad T_n(t) = e^{-t}\sin(\sqrt{n^2-1}\,t) \quad \text{if } n \geq 2.$$

By the superposition principle, a formal solution to (1)-(2)-(3)-(4) is

$$u(x,t) = \sum_{n=0}^{\infty} A_n X_n(x) T_n(t) = A_0(1-e^{-2t}) + A_1\cos(x)te^{-t} + \sum_{n=2}^{\infty} A_n\cos(nx)e^{-t}\sin(\sqrt{n^2-1}\,t)$$

where $A_0, A_1, A_2, \ldots$ are arbitrary constants. Then

$$u_t(x,t) = 2A_0e^{-2t} + A_1\cos(x)(1-t)e^{-t} + \sum_{n=2}^{\infty} A_n\cos(nx)e^{-t}\left[\sqrt{n^2-1}\cos(\sqrt{n^2-1}\,t) - \sin(\sqrt{n^2-1}\,t)\right]$$

so to satisfy (5) we must have

$$x^2 = u_t(x,0) = 2A_0 + A_1\cos(x) + \sum_{n=2}^{\infty} A_n\sqrt{n^2-1}\cos(nx) \quad \text{if } 0 \leq x \leq \pi.$$

By problem 6,

$$2A_0 = \frac{\pi^2}{3}, \quad A_1 = -4, \quad \text{and for } n \geq 2, \quad A_n\sqrt{n^2-1} = \frac{4(-1)^n}{n^2}.$$

Thus a solution to (1)-(2)-(3)-(4)-(5) is

$$\boxed{u(x,t) = \frac{\pi^2}{6}(1-e^{-2t}) - 4\cos(x)te^{-t} + 4e^{-t}\sum_{n=2}^{\infty}\frac{(-1)^n\cos(nx)\sin(\sqrt{n^2-1}\,t)}{n^2\sqrt{n^2-1}}}.$$

5 pts (b) Let $u = u(x,t)$ satisfy (1)-(2)-(3) and consider $E(t) = \frac{1}{2}\int_0^{\pi}\left[u_t^2(x,t) + u_x^2(x,t)\right]dx$, the energy function for u on the interval $0 \leq t < \infty$. We have

$$E'(t) = \frac{1}{2}\int_0^{\pi}\frac{\partial}{\partial t}\left[u_t^2(x,t) + u_x^2(x,t)\right]dx$$

$$= \int_0^{\pi}\left[u_t(x,t)u_{tt}(x,t) + \overbrace{u_x(x,t)}^{v}\overbrace{u_{xt}(x,t)}^{dv}\right]dx$$

$$= \int_0^{\pi} u_t(x,t)u_{tt}(x,t)\,dx + \underbrace{\cancel{u_x(x,t)u_t(x,t)\Big|_{x=0}^{\pi}}}_{0 \text{ by (2)-(3)}} - \int_0^{\pi} u_t(x,t)u_{xx}(x,t)\,dx$$

#7(b) (cont.) $E'(t) = \int_0^\pi u_t(x,t)[u_{tt}(x,t) - u_{xx}(x,t)]dx$

$$= -2\int_0^\pi u_t^2(x,t)dx \qquad (\text{by } (1))$$

$$\leq 0 .$$

Consequently $E = E(t)$ is decreasing on $t \geq 0$.

(c) $\boxed{\text{Yes, there is only one solution to the problem in part (a).}}$ For suppose that $u = v(x,t)$ is another solution to this problem and consider $w(x,t) = u(x,t) - v(x,t)$ for $0 \leq x \leq \pi$ and $t \geq 0$. Then w is continuous and solves

$$(1') \qquad w_{tt} - w_{xx} + 2w_t = 0 \quad \text{if} \quad 0 < x < \pi,\ 0 < t < \infty,$$

$$(2')\text{-}(3') \qquad w_x(0,t) = 0 = w_x(\pi,t) \quad \text{if} \quad t \geq 0,$$

$$(4')\text{-}(5') \qquad w(x,0) = 0 = w_t(x,0) \quad \text{if} \quad 0 \leq x \leq \pi.$$

By part (b), the energy function $E(t) = \frac{1}{2}\int_0^\pi [w_t^2(x,t) + w_x^2(x,t)]dx$ is decreasing on $0 \leq t < \infty$. But then by $(5')$ and differentiation of $(4')$,

$$0 \leq E(t) \leq E(0) = \frac{1}{2}\int_0^\pi [w_t^2(x,0) + w_x^2(x,0)]dx = 0.$$

Consequently the vanishing theorem implies $w_t(x,t) = 0 = w_x(x,t)$ if $0 \leq x \leq \pi$ and $0 \leq t < \infty$, and hence $w(x,t) = \text{constant}$ in the strip $0 \leq x \leq \pi$, $0 \leq t < \infty$. But $(4')$ implies the constant is zero; i.e. $u(x,t) - v(x,t) = w(x,t) = 0$ for all $0 \leq x \leq \pi$ and $0 \leq t < \infty$. Since $v(x,t) = u(x,t)$ in the strip $0 \leq x \leq \pi$, $0 \leq t < \infty$, there is only one solution to the problem in (a).

Math 325

Summer 2012

Final Exam

$n = 16$

mean = 107.9

median = 104

standard deviation = 53.0

Distribution of Scores

Range	Gradute Grade	Undergraduate Grade	Frequency
174 – 200	A	A	3
146 – 173	B	B	2
120 – 145	C	B	0
100 – 119	C	C	3
0 – 99	F	D	8