Mathematics 325
Homework Assignment 2

Due Date: ______________________

Name: Dr. Grow

(a) Verify that $u(x,y)=e^{x-y}$ is a particular solution (i.e. one involving no arbitrary functions) of the nonhomogeneous partial differential equation $yu_x+xu_y=(y-x)e^{x-y}$.

(b) Find the general solution of the homogeneous partial differential equation $yu_x+xu_y=0$.

(c) Find the solution of $yu_x+xu_y=(y-x)e^{x-y}$ that satisfies the auxiliary condition $u(x,0)=x^4+e^x$ for all real x.

(d) What is the largest region of the xy-plane in which the solution in part (c) is uniquely determined?

(a) If $u(x,y)=e^{x-y}$ then $u_x=e^{x-y}$ and $u_y=-e^{x-y}$. Therefore

$$yu_x+xu_y = ye^{x-y}-xe^{x-y} \overset{\checkmark}{=} (y-x)e^{x-y}.$$

(b) The solution to $a(x,y)u_x+b(x,y)u_y=0$ is constant along characteristic curves satisfying $\frac{dy}{dx}=\frac{b(x,y)}{a(x,y)}$. For the PDE in (b) the characteristic curves are given by $\frac{dy}{dx}=\frac{x}{y}$, so separating variables yields $y\,dy=x\,dx$. Integrating once leads to $y^2/2+c_1=x^2/2+c_2$ or $y^2-x^2=c$ where $c=2(c_2-c_1)$ is an arbitrary constant. Along a characteristic curve the solution u is constant:

$$u(x,y)=u(x,\pm\sqrt{c+x^2})=u(0,\pm\sqrt{c})=f(c)$$

where f is an arbitrary differentiable function of a single real variable. Thus $u(x,y)=f(y^2-x^2)$ is the general solution of the (homogeneous linear) PDE in (b).

(c) The general solution of the nonhomogeneous linear PDE in (c) has the form $u(x,y)=u_p(x,y)+u_c(x,y)$ where u_p is any particular solution of the nonhomogeneous PDE in (c) and u_c is the general solution of the associated homogeneous linear PDE in (b). Consequently $u(x,y)=e^{x-y}+f(y^2-x^2)$ is the general solution of the nonhomogeneous linear PDE in (c). (See parts (a) and (b).) We use the auxiliary condition in (c) to determine f:

$$x^4 + e^x = u(x,0) = e^{x-0} + f(0^2 - x^2) = e^x + f(-x^2)$$

for all $-\infty < x < \infty$. Therefore $x^4 = f(-x^2)$ for all real x. Letting $z = -x^2$, this is equivalent to $f(z) = z^2$ for all $z \le 0$. Thus

$$\boxed{u(x,y) = e^{x-y} + (y^2 - x^2)^2}$$

solves the problem in (c).

(d) Since the function f is determined uniquely only for nonpositive arguments ($f(z) = z^2$ for $z \le 0$), it follows that the solution

$$u(x,y) = e^{x-y} + f(y^2 - x^2) = e^{x-y} + (y^2-x^2)^2$$

in (c) is uniquely determined only when $\boxed{y^2 - x^2 \le 0.}$