Ahlfors Chapter 1. Complex Numbers

EXERCISES

pp. 2-3

✓ **1.** Find the values of

$$(1+2i)^3, \qquad \frac{5}{-3+4i}, \qquad \left(\frac{2+i}{3-2i}\right)^2, \qquad (1+i)^n+(1-i)^n.$$

✓ **2.** If $z = x + iy$ (x and y real), find the real and imaginary parts of

$$z^4, \qquad \frac{1}{z}, \qquad \frac{z-1}{z+1}, \qquad \frac{1}{z^2}.$$

3. Show that

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1 \qquad \text{and} \qquad \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$$

for all combinations of signs.

EXERCISES

p. 4

✓ **1.** Compute

$$\sqrt{i}, \qquad \sqrt{-i}, \qquad \sqrt{1+i}, \qquad \sqrt{\frac{1-i\sqrt{3}}{2}}.$$

✓ **2.** Find the four values of $\sqrt[4]{-1}$.

3. Compute $\sqrt[4]{i}$ and $\sqrt[4]{-i}$.

✓ **4.** Solve the quadratic equation

$$z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0.$$

EXERCISES (For students with a background in algebra)

p. 6

✓ **1.** Show that the system of all matrices of the special form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

combined by matrix addition and matrix multiplication, is isomorphic to the field of complex numbers.

2. Show that the complex-number system can be thought of as the field of all polynomials with real coefficients modulo the irreducible polynomial $x^2 + 1$.

pp. 8–9

EXERCISES

1. Verify by calculation that the values of

$$\frac{z}{z^2+1}$$

for $z = x + iy$ and $z = x - iy$ are conjugate.

2. Find the absolute values of

$$-2i(3+i)(2+4i)(1+i) \qquad \text{and} \qquad \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}.$$

3. Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| = 1$$

if either $|a| = 1$ or $|b| = 1$. What exception must be made if $|a| = |b| = 1$?

4. Find the conditions under which the equation $az + b\bar{z} + c = 0$ in one complex unknown has exactly one solution, and compute that solution.

5. Prove Lagrange's identity in the complex form

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{1 \le i < j \le n} |a_i\bar{b}_j - a_j\bar{b}_i|^2.$$

pp. 11–12

EXERCISES

1. Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1$$

if $|a| < 1$ and $|b| < 1$.

2. Prove Cauchy's inequality by induction.

3. If $|a_i| < 1$, $\lambda_i \geqq 0$ for $i = 1, \ldots, n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, show that

$$|\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n| < 1.$$

4. Show that there are complex numbers z satisfying

$$|z - a| + |z + a| = 2|c|$$

if and only if $|a| \leqq |c|$. If this condition is fulfilled, what are the smallest and largest values of $|z|$?

EXERCISES

p. 15

1. Find the symmetric points of a with respect to the lines which bisect the angles between the coordinate axes.

2. Prove that the points a_1, a_2, a_3 are vertices of an equilateral triangle if and only if $a_1^2 + a_2^2 + a_3^2 = a_1a_2 + a_2a_3 + a_3a_1$.

3. Suppose that a and b are two vertices of a square. Find the two other vertices in all possible cases.

4. Find the center and the radius of the circle which circumscribes the triangle with vertices a_1, a_2, a_3. Express the result in symmetric form.

EXERCISES

pp. 16-17

1. Express $\cos 3\varphi$, $\cos 4\varphi$, and $\sin 5\varphi$ in terms of $\cos \varphi$ and $\sin \varphi$.

2. Simplify $1 + \cos \varphi + \cos 2\varphi + \cdots + \cos n\varphi$ and $\sin \varphi + \sin 2\varphi + \cdots + \sin n\varphi$.

3. Express the fifth and tenth roots of unity in algebraic form.

4. If ω is given by (23), prove that

$$1 + \omega^h + \omega^{2h} + \cdots + \omega^{(n-1)h} = 0$$

for any integer h which is not a multiple of n.

EXERCISES

p. 17

1. When does $az + b\bar{z} + c = 0$ represent a line?

2. Write the equation of an ellipse, hyperbola, parabola in complex form.

3. Prove that the diagonals of a parallelogram bisect each other and that the diagonals of a rhombus are orthogonal.

4. Prove analytically that the midpoints of parallel chords to a circle lie on a diameter perpendicular to the chords.

5. Show that all circles that pass through a and $1/\bar{a}$ intersect the circle $|z| = 1$ at right angles.

EXERCISES

p. 20

1. Show that z and z' correspond to diametrically opposite points on the Riemann sphere if and only if $z\bar{z}' = -1$.

2. A cube has its vertices on the sphere S and its edges parallel to the coordinate axes. Find the stereographic projections of the vertices.

3. Same problem for a regular tetrahedron in general position.

4. Let Z, Z' denote the stereographic projections of z, z', and let N be the north pole. Show that the triangles NZZ' and Nzz' are similar, and use this to derive (28).

5. Find the radius of the spherical image of the circle in the plane whose center is a and radius R.