Mathematics 351 | Final Exam Fall 2006 | Name:______________________ (4 points)

All problems on this examination are of equal value: 28 points.

Part A. Solve any three of the following four problems. Circle the numbers of those three problems in this part which you want me to grade.

A1. Represent graphically the set of complex numbers z for which $\left|\dfrac{z-3}{z+3}\right| < 2.$

A2. (a) Show that $\psi(x,y) = \ln\left[(x-1)^2 + (y-2)^2\right]$ is harmonic in every region which does not contain the point (1,2).

(b) Find a function $\phi = \phi(x,y)$ such that $\phi + i\psi$ is analytic.

(c) Express $\phi + i\psi$ as a function of $z = x + iy$.

A3. Find a Laurent series centered at $z = 1$ for $F(z) = \dfrac{3z-3}{(2z-1)(z-2)}$ and determine the set of $z-$values for which this series is convergent.

A4. Find a linear fractional transformation which maps the three points $0,\ -i,\ -1$ in the $z-$plane into $i,\ 1,\ 0$ in the $w-$plane, respectively.

Part B. Solve any four of the following five problems. Circle the numbers of those four problems in this part which you want me to grade.

B1. Show that $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(k\theta)}{2^k} = \frac{2\sin(\theta)}{5 - 4\cos(\theta)}$ for all real numbers $\theta.$

B2. State and prove the Cauchy-Goursat Theorem for rectangles.

B3. Suppose that $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ converges for all $|z| < 1.$ If $0 \le r < 1,$ show that

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

B4. If $m > 0,$ show that $\displaystyle\int_0^{\infty} \frac{\cos(mx)}{x^2+1}\,dx = \frac{\pi}{2e^m}.$

B5. Find the bounded electrostatic potential $V = V(x,y)$ in the space $y > 0$ bounded by an infinite conducting plane $y = 0,$ one strip ($-a < x < a,\ y = 0$) of which is insulated from the rest of the plane and kept at potential $V = 1,$ while $V = 0$ on the rest ($|x| > a,\ y = 0$).

Bonus: In B3, find, with proof, hypotheses which will ensure that $F(e^{i\theta}) = \lim\limits_{r\to 1^-} f(re^{i\theta})$ exists for almost every θ in $[0, 2\pi)$ and $\displaystyle\frac{1}{2\pi}\int_0^{2\pi} \left|F\left(e^{i\theta}\right)\right|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2.$