Math 5351

Homework 4

- p. 44: # 1, 2, 3, 4

- p. 47: # 3, 4, 5, 6, 8

#1, p.44: Find the values of $\sin(i)$, $\cos(i)$, $\tan(1+i)$.

Using the identity $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$ with $z = i$, we obtain

$$\sin(i) = \frac{e^{i(i)} - e^{-i(i)}}{2i} = \frac{-i}{2}\left(e^{-1} - e\right) = \boxed{\frac{i}{2}\left(e - \frac{1}{e}\right)} \doteq i(1.1752).$$

Similarly, using the identity $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$ with $z = i$, we have

$$\cos(i) = \frac{e^{i(i)} + e^{-i(i)}}{2} = \frac{1}{2}\left(e^{-1} + e^{1}\right) = \boxed{\frac{1}{2}\left(e + \frac{1}{e}\right)} \doteq 1.5431.$$

Note: These values can be expressed as $\boxed{\sin(i) = i\sinh(1)}$ and $\boxed{\cos(i) = \cosh(1)}$. (cf. #2, p.44.)

Using the identity $\tan(z) = -i\left(\dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right)$ with $z = 1+i$ gives

$$\tan(1+i) = -i\left(\frac{e^{i(1+i)} - e^{-i(1+i)}}{e^{i(1+i)} + e^{-i(1+i)}}\right)$$

$$= -i\left(\frac{e^{-1}e^{i} - ee^{-i}}{e^{-1}e^{i} + ee^{-i}}\right)$$

$$= -i\left(\frac{e^{-1}e^{i} - ee^{-i}}{e^{-1}e^{i} + ee^{-i}}\right)\left(\frac{e^{-1}e^{-i} + ee^{i}}{e^{-1}e^{-i} + ee^{i}}\right)$$

$$= -i\left(\frac{e^{-2} - e^{-2i} + e^{2i} - e^{2}}{e^{-2} + e^{-2i} + e^{2i} + e^{2}}\right)$$

$$= -i\left(\frac{e^{-2} - e^{2} + 2i\sin(2)}{e^{2} + e^{-2} + 2\cos(2)}\right)$$

$$= \boxed{\frac{2\sin(2) + i(e^{2} - e^{-2})}{2\cos(2) + e^{2} + e^{-2}}}.$$

(cont.)

This can be expressed as

$$\tan(1+i) = \frac{2\sin(2) + 2i\sinh(2)}{2\cos(2) + 2\cosh(2)}$$

$$= \boxed{\frac{\sin(2) + i\sinh(2)}{\cos(2) + \cosh(2)}} \, .$$

(Cf. #2, p. 44.)

#2, p. 44 The hyperbolic cosine and sine are defined by $\cosh(z) = (e^z + e^{-z})/2$, $\sinh(z) = (e^z - e^{-z})/2$. Express them through $\cos(iz)$, $\sin(iz)$. Derive the addition formulas, and formulas for $\cosh(2z)$, $\sinh(2z)$.

$$\cosh(z) = \frac{1}{2}(e^z + e^{-z}) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{n!} + \frac{(-1)^n}{n!}\right) z^n.$$

But
$$1 + (-1)^n = \begin{cases} 1+1 & \text{if } n \text{ is even,} \\ 1-1 & \text{if } n \text{ is odd,} \end{cases} = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

$$\therefore \boxed{\cosh(z) =} \frac{1}{2}\sum_{k=0}^{\infty} \frac{2z^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k (iz)^{2k}}{(2k)!} = \boxed{\cos(iz)}.$$

$$\sinh(z) = \frac{1}{2}(e^z - e^{-z}) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1-(-1)^n}{n!}\right) z^n.$$

As above, $1 - (-1)^n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$ Therefore

$$\boxed{\sinh(z) =} \frac{1}{2}\sum_{k=0}^{\infty} \frac{2z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{-i(-1)^k (iz)^{2k+1}}{(2k+1)!} = \boxed{-i\sin(iz)}.$$

Using these identities for $\cosh(z)$ and $\sinh(z)$ and the addition formulas for cosine and sine gives the identities:

$$\begin{aligned}\boxed{\cosh(a+b) =} \cos(i(a+b)) &= \cos(ia)\cos(ib) - \sin(ia)\sin(ib) \\ &= \cosh(a)\cosh(b) + (-i\sin(ia))(-i\sin(ib)) \\ &= \boxed{\cosh(a)\cosh(b) + \sinh(a)\sinh(b)}\end{aligned}$$

and

$$\boxed{\sinh(a+b) =} -i\sin(i(a+b)) = -i\left[\sin(ia)\cos(ib) + \cos(ia)\sin(ib)\right]$$

$$= (-i\sin(ia))\cos(ib) + \cos(ia)(-i\sin(ib))$$

$$= \boxed{\sinh(a)\cosh(b) + \cosh(a)\sinh(b)}.$$

Consequently, taking $a = b = z$, we have the identities

$$\boxed{\cosh(2z) = \cosh^2(z) + \sinh^2(z)}$$

and

$$\boxed{\sinh(2z) = 2\sinh(z)\cosh(z)}.$$

#3, p.44: Use the addition formulas to separate $\cos(x+iy)$, $\sin(x+iy)$ in real and imaginary parts.

(cf. #2, p.44)

Using $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, $\cos(iz) = \cosh(z)$, and $-i\sin(iz) = \sinh(z)$ we find

$$\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) = \boxed{\underbrace{\cos(x)\cosh(y)}_{Re(\cos(x+iy))} + i\underbrace{[-\sin(x)\sinh(y)]}_{Im(\cos(x+iy))}}.$$

Using $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$,

$$\sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy) = \boxed{\underbrace{\sin(x)\cosh(y)}_{Re(\sin(x+iy))} + i\underbrace{\cos(x)\sinh(y)}_{Im(\sin(x+iy))}}$$

#4, p. 44: Show that

$$|\cos(z)|^2 \overset{1}{=} \sinh^2(y) + \cos^2(x) \overset{2}{=} \cosh^2(y) - \sin^2(x) \overset{3}{=} \frac{1}{2}\left(\cosh(2y) + \cos(2x)\right)$$

and

$$|\sin(z)|^2 \overset{4}{=} \sinh^2(y) + \sin^2(x) \overset{5}{=} \cosh^2(y) - \cos^2(x) \overset{6}{=} \frac{1}{2}\left(\cosh(2y) - \cos(2x)\right).$$

From #3, p. 44 we have

$$(*) \quad |\cos(z)|^2 = |\cos(x+iy)| = \left|\cos(x)\cosh(y) + i\left[-\sin(x)\sinh(y)\right]\right|^2 = \cos^2(x)\cosh^2 y + \sin^2(x)\sinh^2(y).$$

But $\cosh^2(y) - \sinh^2(y) = 1$ and $\cos^2(x) + \sin^2(x) = 1$ for all real x and y so

$$(**) \quad \cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y) = \cos^2(x)(1 + \sinh^2(y)) + \sin^2(x)\sinh^2(y)$$
$$= \cos^2(x) + (\cos^2(x) + \sin^2(x))\sinh^2(y)$$
$$= \cos^2(x) + \sinh^2(y).$$

From $(*)$ and $(**)$ we obtain 1. Also

$$(***) \quad \cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y) = (1 - \sin^2(x))\cosh^2(y) + \sin^2(x)\sinh^2(y)$$
$$= \cosh^2(y) + \sin^2(x)(\sinh^2(y) - \cosh^2(y))$$
$$= \cosh^2(y) - \sin^2(x).$$

From $(**)$ and $(***)$ we obtain 2. Using the double "angle" identities $\cos(2x) = \cos^2(x) - \sin^2(x)$ and $\cosh(2y) = \cosh^2(y) + \sinh^2(y)$ (cf. #2, p. 44) it follows that

$$\frac{1}{2}\Big(\cosh(2y)+\cos(2x)\Big) = \frac{1}{2}\Big(\cosh^2(y)+\sinh^2(y)+\cos^2(x)-\sin^2(x)\Big)$$

$$= \frac{1}{2}\Big(\cosh^2(y)+(-1+\cosh^2 y)+(1-\sin^2(x))-\sin^2(x)\Big)$$

$$= \cosh^2(y)-\sin^2(x),$$

which establishes ③.

From #3, p. 44 we have

$$(\dagger) \qquad |\sin(z)|^2 = |\sin(x+iy)|^2 = |\sin(x)\cosh(y)+i\cos(x)\sinh(y)|^2 = \sin^2(x)\cosh^2(y)+\cos^2(x)\sinh^2(y).$$

Using the identities $\cosh^2(y)-\sinh^2(y)=1$ and $\cos^2(x)+\sin^2(x)=1$, it follows that

$$(\dagger\dagger) \qquad \sin^2(x)\cosh^2(y)+\cos^2(x)\sinh^2(y) = \sin^2(x)(1+\sinh^2(y))+\cos^2(x)\sinh^2(y)$$

$$= \sin^2(x)+(\sin^2(x)+\cos^2(x))\sinh^2(y)$$

$$= \sin^2(x)+\sinh^2(y).$$

We obtain ④ from $(\dagger)$ and $(\dagger\dagger)$. Similarly,

$$(\dagger\dagger\dagger) \qquad \sin^2(x)\cosh^2(y)+\cos^2(x)\sinh^2(y) = (1-\cos^2(x))\cosh^2(y)+\cos^2(x)\sinh^2(y)$$

$$= \cosh^2(y)+\cos^2(x)(\sinh^2(y)-\cosh^2(y))$$

$$= \cosh^2(y)-\cos^2(x)$$

and ⑤ follows from $(\dagger\dagger)$ and $(\dagger\dagger\dagger)$. Finally, the double "angle" identities show

$$\frac{1}{2}\Big(\cosh(2y)-\cos(2x)\Big) = \frac{1}{2}\Big(\cosh^2(y)+\sinh^2(y)-\cos^2(x)+\sin^2(x)\Big)$$

$$= \frac{1}{2}\Big(\cosh^2(y)+(\cosh^2(y)-1)-\cos^2(x)+(1-\cos^2(x))\Big)$$

$$= \cosh^2(y)-\cos^2(x)$$

which verifies ⑥.

#3, p.47: Find the value of e^{z} for $z = \frac{-\pi i}{2}, \frac{3\pi i}{4}, \frac{2\pi i}{3}$.

We use the identity $e^{iz} = \cos(z) + i\sin(z)$.

$$e^{\frac{-\pi i}{2}} = \cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right) = \boxed{-i}$$

$$e^{\frac{3\pi i}{4}} = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) = \boxed{-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}}$$

$$e^{\frac{2\pi i}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \boxed{\frac{-1}{2} + i\frac{\sqrt{3}}{2}}$$

#4, p. 47: For what values of z is e^z equal to $2, -1, i, \frac{-i}{2}, -1-i, 1+2i$?

Principle: If $w \neq 0$ then $e^{x+iy} = w$ if and only if $e^x = |w|$ and $e^{iy} = \frac{w}{|w|}$.

$$e^{x+iy} = 2 \iff e^x = 2 \text{ and } e^{iy} = 1$$

$$\iff x = \ln(2) \text{ and } y = 2\pi k \text{ where } k \in \mathbb{Z}.$$

Therefore $e^z = 2$ implies $z = x+iy = \boxed{\ln(2) + 2\pi i k}$ where $k = 0, \pm 1, \pm 2, \ldots$

$$e^{x+iy} = -1 \iff e^x = |-1| \text{ and } e^{iy} = -1$$

$$\iff x = 0 \quad \text{and} \quad y = \pi + 2k\pi \quad \text{where } k \in \mathbb{Z}.$$

Thus $e^{x+iy} = -1$ implies $z = x+iy = \boxed{i(2k+1)\pi}$ where $k \in \mathbb{Z}$.

$$e^{x+iy} = i \iff e^x = |i| = 1 \text{ and } e^{iy} = \frac{i}{|i|} = i$$

$$\iff x = 0 \quad \text{and} \quad y = \frac{\pi}{2} + 2k\pi \quad (k \in \mathbb{Z}).$$

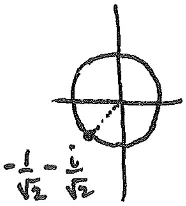
Thus $e^z = i$ implies $z = x+iy = \boxed{i(4k+1)\frac{\pi}{2}}$ $(k = 0, \pm 1, \pm 2, \ldots)$.

$$e^{x+iy} = \frac{-i}{2} \iff e^x = \left|\frac{-i}{2}\right| = \frac{1}{2} \text{ and } e^{iy} = \frac{-i/2}{|-i/2|} = -i$$

$$\iff x = \ln\frac{1}{2} = -\ln 2 \quad \text{and} \quad y = \frac{-\pi}{2} + 2k\pi.$$

Thus $e^z = \frac{-i}{2}$ implies $z = x+iy = \boxed{-\ln(2) + i(4k-1)\frac{\pi}{2}}$ $(k = 0, \pm 1, \pm 2, \ldots)$.

$$e^{x+iy} = -1-i \iff e^{x} = |-1-i| = \sqrt{2} \quad \text{and} \quad e^{iy} = \frac{-1-i}{|-1-i|} = -\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$



$$\iff x = \ln\sqrt{2} = \frac{1}{2}\ln 2 \quad \text{and} \quad y = \frac{-3\pi}{4} + 2k\pi \quad (k \in \mathbb{Z})$$

Thus $e^{z} = -1-i$ implies $z = x+iy = \boxed{\frac{1}{2}\ln(2) + i(8k-3)\frac{\pi}{4}}$ $(k = 0, \pm 1, \pm 2, \ldots)$.

$$e^{x+iy} = 1+2i \iff e^{x} = |1+2i| = \sqrt{5} \quad \text{and} \quad e^{iy} = \frac{1+2i}{|1+2i|} = \frac{1}{\sqrt{5}} + i\frac{2}{\sqrt{5}}.$$

$$\iff x = \ln\sqrt{5} = \frac{1}{2}\ln(5) \quad \text{and} \quad y = \operatorname{Arccos}\left(\frac{1}{\sqrt{5}}\right) + 2k\pi$$

Thus $e^{z} = 1+2i$ implies $z = \boxed{\frac{1}{2}\ln(5) + i\left(\operatorname{Arccos}\left(\frac{1}{\sqrt{5}}\right) + 2k\pi\right)}$ $(k = 0, \pm 1, \pm 2, \ldots)$.

#5, p.47: Find the real and imaginary parts of $\exp(e^{z})$.

If $w = u+iv$ where u and v are real then

$$e^{w} = e^{u+iv} = e^{u}e^{iv} = e^{u}(\cos(v)+i\sin(v)) = e^{u}\cos(v)+ie^{u}\sin(v).$$

That is, $\mathrm{Re}(e^{u+iv}) = e^{u}\cos(v)$ and $\mathrm{Im}(e^{u+iv}) = e^{u}\sin(v)$.

Now $\exp(e^{z}) = \exp(e^{x+iy}) = \exp(e^{x}\cos(y)+ie^{x}\sin(y))$ so

$$\mathrm{Re}(\exp(e^{x+iy})) = \boxed{e^{e^{x}\cos(y)}\cos(e^{x}\sin(y))}$$

and

$$\mathrm{Im}(\exp(e^{x+iy})) = \boxed{e^{e^{x}\cos(y)}\sin(e^{x}\sin(y))}.$$

Note: These expressions can be represented directly in terms of z via the substitutions $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$.

#6, p. 47 Determine all values of 2^i, i^i, $(-1)^{2i}$.

$$\boxed{2^i =} e^{i\log 2} = e^{i\ln 2} = \boxed{\cos(\ln 2) + i\sin(\ln 2)}.$$

$$\boxed{i^i =} e^{i\log i} = e^{i(\ln|i| + i\arg i)} = e^{-\arg i} = \boxed{\left\{ e^{-\frac{\pi}{2} - 2k\pi} : k = 0, \pm 1, \pm 2, \dots \right\}}.$$

$$\boxed{(-1)^{2i} =} e^{2i\log(-1)} = e^{2i(\ln|-1| + i\arg(-1))} = e^{-2\arg(-1)} = \boxed{\left\{ e^{-2(2k+1)\pi} : k = 0, \pm 1, \pm 2, \dots \right\}}.$$

Note: We are using the definition

$$a^b = e^{b\log(a)} = \begin{cases} e^{b\ln(a)} & \text{if } a > 0, \\ e^{b(\ln|a| + i\arg(a))} & \text{if } a \neq 0 \text{ and } a \text{ not pos.} \end{cases}$$

#8, p. 47 : Express arctan w in terms of the logarithm.

$z = \arctan(w)$ is equivalent to $w = \tan(z) = -i\left(\dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right)$,

where we have used have used the identity for tangent on page 44.

Solving for z in terms of w gives

$$(e^{iz} + e^{-iz})(iw) = e^{iz} - e^{-iz}$$

$$e^{iz}(e^{iz} + e^{-iz})(iw) = e^{iz}(e^{iz} - e^{-iz})$$

$$ie^{2iz}w + iw = e^{2iz} - 1$$

$$1 + iw = e^{2iz}(1 - iw)$$

$$\frac{i}{i}\left(\frac{1+iw}{1-iw}\right) = e^{2iz}$$

$$\log\left(\frac{i-w}{i+w}\right) = 2iz .$$

$$\therefore \quad \boxed{\arctan(w) = z = \frac{1}{2i}\log\left(\frac{i-w}{i+w}\right)} .$$