Math 5351

Homework 5

- Lecture Problem 1
- p. 108: # 1, 2, 3, 4, 5, 7

Lecture Problem 1

(a) Show that $f(z) = \widetilde{\text{Log}}(1+z)$ is analytic in the region

$$R = \mathbb{C} \setminus \{ z = x+iy : x \le -1 \text{ and } y = 0 \}$$

with derivative $f'(z) = \dfrac{1}{1+z}$ for $z \in R$.

(b) Show that the function $g(z) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$ is analytic in the open disk $D(0;1) = \{ z \in \mathbb{C} : |z| < 1 \}$ and $g'(z) = \dfrac{1}{1+z}$ if $z \in D(0;1)$.

(c) Use (a) and (b) to show that $\widetilde{\text{Log}}(1+z) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$ if $z \in D(0;1)$.

(d) Use (c) to show that $-\widetilde{\text{Log}}(1-z) = \displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$ if $z \in D(0;1)$.

(e) Use (d), continuity of the principal logarithm function, and Abel's Limit Theorem (Theorem 3, p.41) to conclude that

$$-\widetilde{\text{Log}}(1 - e^{i\theta}) = \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n} = \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} + i \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n}$$

for all $0 < \theta < 2\pi$. (Hint: What have we shown about the sequence of partial sums of the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n}$?)

(f) Use $\widetilde{\text{Log}}(w) = \ln|w| + i\widetilde{\text{Arg}}(w)$ and (e) to show that

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} = \ln\left(\frac{1}{2\sin(\theta/2)} \right) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} = \frac{\pi - \theta}{2}$$

for all $0 < \theta < 2\pi$.

(g) Find the sums of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$.

Part(a): $g(w) = \widetilde{\operatorname{Log}}\, w$ is analytic in the region

$$\mathbb{C} \setminus \{ w = u+iv : \ u \le 0 \text{ and } v = 0 \},$$

with derivative $g'(w) = \frac{1}{w}$. The function $w = h(z) = 1+z$ is analytic in the entire complex plane with derivative $h'(z) = 1$ and maps $\mathbb{C} \setminus \{ z = x+iy : \ x \le -1 \text{ and } y = 0 \}$ onto $\mathbb{C} \setminus \{ w = u+iv : \ u \le 0 \text{ and } v = 0 \}$. Consequently the composition $f(z) = g(h(z))$ is analytic in the region

$$R = \mathbb{C} \setminus \{ z = x+iy : \ x \le -1 \text{ and } y = 0 \}$$

with derivative

$$f'(z) = g'(h(z))h'(z) = \frac{1}{1+z}.$$

Part(b): The function $g(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$, being a power series with radius of convergence

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n-1}/n}{(-1)^n/n+1} \right| = \lim_{n\to\infty} \frac{n+1}{n} = 1,$$

is analytic in its disk of convergence $D(0;1)$. Furthermore, its derivative can be obtained through term-by-term differentiation:

$$g'(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n z^{n-1}}{n} = \sum_{n=1}^{\infty} (-z)^{n-1} = \frac{1}{1-(-z)} = \frac{1}{1+z},$$

the third equality following from a geometric series, convergent for $|z|<1$.

Part (c): Consider the function F defined in $D(0;1)$ by

$$F(z) = \widetilde{\operatorname{Log}}(1+z) - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n .$$

Since $z \mapsto \widetilde{\operatorname{Log}}(1+z)$ is analytic in $\mathbb{C} \setminus \{z = x+iy : x \le -1 \text{ and } y = 0\}$ by part (a) and $z \mapsto \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$ is analytic in $D(0;1)$ by part (b), it follows that F is analytic in $D(0;1)$ with derivative

$$F'(z) = \frac{1}{1+z} - \frac{1}{1+z} = 0$$

for $|z| < 1$. Consequently F is a constant function in $D(0;1)$. But clearly $F(0) = 0$. Hence $F(z) = 0$ for all z in $D(0;1)$; i.e.

$$\widetilde{\operatorname{Log}}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n \qquad (z \in D(0;1)).$$

Part (d): Notice that z belongs to $D(0;1)$ if and only if $-z$ belongs to $D(0;1)$. Therefore, replacing z by $-z$ in the identity from part (c), we obtain

$$\widetilde{\operatorname{Log}}(1-z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-z)^n}{n} = -\sum_{n=1}^{\infty} \frac{z^n}{n}$$

for all $z \in D(0;1)$. This is equivalent to the desired identity.

Part (e): We will use the following version of Abel's limit theorem (Theorem 3 on page 41):

If $\sum_{n=0}^{\infty} a_n e^{in\theta}$ converges for some $\theta \in [0, 2\pi]$ then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ tends to $f(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta}$ as z approaches $e^{i\theta}$ from inside the unit disk in such a way that $\dfrac{|e^{i\theta} - z|}{1 - |z|}$ remains bounded.

Recall that we have shown (the sequence of partial sums of) the series $\sum_{n=1}^{\infty} \frac{e^{in\theta}}{n}$ converges for all $\theta \in (0, 2\pi)$. For all such θ we have

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} + i \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} = \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n}$$

$$= \lim_{r \to 1^-} \sum_{n=1}^{\infty} \frac{1}{n}(re^{i\theta})^n \qquad \text{(by Abel's limit theorem)}$$

$$= \lim_{r \to 1^-} -\widetilde{\mathrm{Log}}(1 - re^{i\theta}) \qquad \text{(by part (d))}$$

$$= -\widetilde{\mathrm{Log}}(1 - e^{i\theta}). \qquad \left(\text{by continuity of } w \mapsto \widetilde{\mathrm{Log}}(w) \text{ in } \mathbb{C} \setminus \{w = u + iv : u \le 0, v = 0\}\right)$$

<u>Part (f)</u>: If $0 < \theta < 2\pi$ then

$$\widetilde{\mathrm{Log}}(1 - e^{i\theta}) = \ln|1 - e^{i\theta}| + i\,\widetilde{\mathrm{Arg}}(1 - e^{i\theta}).$$

Note that $1 - e^{i\theta} = 1 - \cos\theta - i\sin\theta$ so $|1 - e^{i\theta}| = \sqrt{(1-\cos\theta)^2 + (-\sin\theta)^2}$

$$= \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta} = \sqrt{2 - 2\cos\theta} = \sqrt{4\sin^2(\theta/2)} = 2\sin(\theta/2).$$

Also $\widetilde{\text{Arg}}(1-e^{i\theta}) = y$ if and only if $e^{iy} = \dfrac{1-e^{i\theta}}{|1-e^{i\theta}|}$ and $-\pi < y \le \pi$.

But $$\frac{1-e^{i\theta}}{|1-e^{i\theta}|} = \frac{(1-\cos\theta) - i\sin\theta}{2\sin(\theta/2)} = \frac{2\sin^2(\theta/2) - i\,2\sin(\theta/2)\cos(\theta/2)}{2\sin(\theta/2)}$$

$$= \sin(\theta/2) - i\cos(\theta/2) = -i\left(\cos(\theta/2) + i\sin(\theta/2)\right) = e^{-i\frac{\pi}{2}} \cdot e^{\frac{i\theta}{2}} = e^{i\left(\frac{\theta-\pi}{2}\right)}.$$

observe that $0 < \theta < 2\pi$ implies $-\frac{\pi}{2} < \frac{\theta-\pi}{2} < \frac{\pi}{2}$. Therefore

$$\widetilde{\text{Arg}}(1-e^{i\theta}) = y = \frac{\theta-\pi}{2}.$$

Consequently, for all $\theta \in (0, 2\pi)$,

$$-\widetilde{\text{Log}}(1-e^{i\theta}) = -\ln|1-e^{i\theta}| - i\,\widetilde{\text{Arg}}(1-e^{i\theta})$$

$$= -\ln(2\sin(\theta/2)) - i\left(\frac{\theta-\pi}{2}\right)$$

$$= \ln\left(\frac{1}{2\sin(\theta/2)}\right) + i\left(\frac{\pi-\theta}{2}\right).$$

Comparing this last identity with the identity in part(e), we conclude that

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} = \ln\left(\frac{1}{2\sin(\theta/2)}\right) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} = \frac{\pi-\theta}{2}$$

for all $0 < \theta < 2\pi$.

<u>Part(g)</u>: Taking $\theta = \pi$ in the identity $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n} = \ln\left(\frac{1}{2\sin(\theta/2)}\right)$

yields $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln\left(\frac{1}{2}\right) = \boxed{-\ln(2).}$$

We take $\theta = \pi/2$ in the identity $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n} = \frac{\pi-\theta}{2} \quad (0<\theta<2\pi)$

to obtain

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

or

$$\boxed{\frac{\pi}{4}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} .$$

#2, p. 108: Compute $\int_{|z|=r} x\,dz$, for the positive sense of the circle, in two ways: first, by use of a parameter, and second, by observing that $x = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}(z + \frac{r^2}{z})$ on the circle.

First way: We parametrize the circle $|z|=r$ by $z(\theta)=re^{i\theta}$ $(0 \le \theta \le 2\pi)$ so $dz = ire^{i\theta}d\theta$. Hence

$$\int_{|z|=r} x\,dz = \int_0^{2\pi} r\cos(\theta)\, ire^{i\theta}d\theta = ir^2\int_0^{2\pi}(\cos^2\theta + i\cos\theta\sin\theta)d\theta$$

$$= ir^2\left[\int_0^{2\pi} \frac{1+\cos(2\theta)}{2}d\theta + i\int_0^{2\pi}\sin(\theta)d(\sin\theta)\right] = ir^2\left[\pi + \frac{i\sin^2\theta}{2}\Big|_0^{2\pi}\right] = \boxed{ir^2\pi}.$$

Second way: Substituting $x = \frac{1}{2}(z+\frac{r^2}{z})$ for $|z|=r$ and using the fact that $\int_{|z|=r} z^m dz = \begin{cases} 2\pi i & \text{if } m=-1, \\ 0 & \text{if } m\neq -1, \end{cases}$ we have

$$\int_{|z|=r} x\,dz = \int_{|z|=r}\frac{1}{2}\left(z+\frac{r^2}{z}\right)dz = \frac{1}{2}\int_{|z|=r} z\,dz + \frac{r^2}{2}\int_{|z|=r}\frac{dz}{z} = 0 + \frac{r^2}{2}(2\pi i)$$

$$= \boxed{ir^2\pi}.$$

#7, p.108 If $P = P(z)$ is a polynomial and C denotes the circle $|z-a| = R$, what is the value of $\int_C P(z)\,d\bar{z}$?
(Answer: $-2\pi i R^2 \overline{P'(a)}$.)

By definition (cf. p.103), $\int_C P(z)\,d\bar{z} = \overline{\int_C \overline{P(z)}\,dz}$. Now let P be a polynomial function of degree $n \geq 0$ and expand P in a (finite) Taylor series about a:

$$P(z) = \sum_{k=0}^{n} \frac{P^{(k)}(a)}{k!}(z-a)^k.$$

Since the circle $C: |z-a| = R$ has parametrization $z(\theta) = a + Re^{i\theta}$ $(0 \leq \theta \leq 2\pi)$ with $dz = iRe^{i\theta}d\theta$,

$$\int_C \overline{P(z)}\,dz = \sum_{k=0}^{n} \frac{\overline{P^{(k)}(a)}}{k!} \int_0^{2\pi} \overline{(Re^{i\theta})^k}\, iRe^{i\theta}d\theta$$

$$= \sum_{k=0}^{n} \frac{iR^{k+1}\overline{P^{(k)}(a)}}{k!} \int_0^{2\pi} e^{i(1-k)\theta}\,d\theta.$$

But $\int_0^{2\pi} e^{im\theta}\,d\theta = \begin{cases} 2\pi & \text{if } m=0, \\ 0 & \text{if } m \neq 0, \end{cases}$ and therefore $\int_C \overline{P(z)}\,dz = 2\pi i R^2 \overline{P'(a)}$.

Consequently

$$\int_C P(z)\,d\bar{z} = \overline{\int_C \overline{P(z)}\,dz} = \overline{2\pi i R^2 \overline{P'(a)}} = -2\pi i R^2 P'(a).$$