Anticipations of Calculus - Archimedes

> Let ABC be a segment of a parabola bounded by the straight line AC and the parabola ABC, and let D be the middle point of AC. Draw the straight line DBE parallel to the axis of the parabola and join AB, BC. Then shall the segment ABC be $\frac{4}{3}$ of the triangle ABC.
>
> Proposition 1 from the *Method* of Archimedes.

The greatest mathematician of antiquity was Archimedes of Syracuse, who lived in the third century B.C. His work on areas of certain curvilinear plane figures and on the areas and volumes of certain curved surfaces used methods that came close to modern integration. One of the characteristics of the ancient Greek mathematicians is that they published their theormes as finished masterpieces, with no hint of the method by which they were evolved. While this makes for beautiful mathematics, it precludes much insight into their methods of discovery. An exception to this state of affairs is Archimedes' *Method*, a work addressed to his friend Eratosthenes, which was known only by references to it until its rediscovery in 1906 in Constantinople by the German mathematical historian J. L. Heiberg. In the *Method*, Archimedes describes how he investigated certain theorems and became convinced of their truth, but he was careful to point out that these investigations did not constitute rigorous proofs of the theorems. In his own (translated) words: "Now the fact here stated is not actually demonstrated by the argument used; but that argument has given a sort of indication that the conclusion is true. Seeing then that the theorem is not demonstrated, but at the same time suspecting that the conclusion is true, we shall have recourse to the geometrical demonstartion which I myself discovered and have already published." This section will give both Archimedes' investigations, from the *Method*, and the rigorous proof, from his *Quadrature of the Parabola*, of the proposition above. The arguments given below are from T. L. Heath's *The Works of Archimdes*, which is a translation "edited in modern notation".

Proposition 1 from the *Method* is stated at the beginning of this section, and the following investigation refers to Figure 1.

> From A draw AKF parallel to DE, and let the tangent to the parabola at C meet DBE in E and AKF in F. Produce CB to meet AF in K, and again produce CK to H, making KH equal to CK.

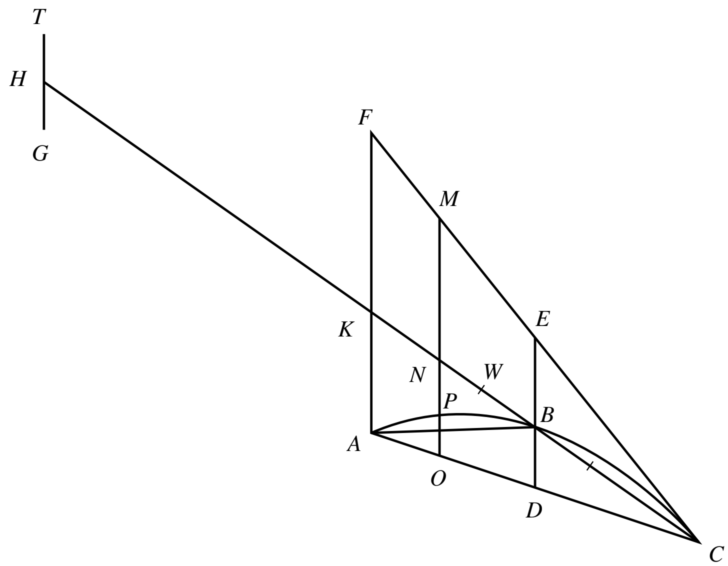


Figure 1:

Consider CH as the bar of a balance, K being its middle point.

Let MO be any straight line parallel to ED, and let it meet CF, CK, AC in M, N, O and the curve in P.

Now, since CE is a tangent to the parabola and CD the semi-ordinate,

$$EB = BD;$$

"for this is proved in the Elements (of Conics)." (by Aristaeus & Euclid)

Since FA, MO, are parallel to ED, it follows that

$$FK = KA, \; MN = NO.$$

Now, by that property of the parabola, "proved in a lemma,"

$$\begin{aligned} MO : OP &= CA : AO \quad \text{(Cf. \textit{Quadrature of the Parabola}, Prop. 5)} \\ &= CK : KN \quad \text{(Euclid, VI. 2)} \\ &= HK : KN. \end{aligned}$$

Take a straight line TG equal to OP, and place it with its centre of gravity at H, so that $TH = HG$; then, since N is the centre of gravity of the straight line MO, and $MO : TG = HK : KN$, it follows that TG at H and MO at N will be in equilibrium about K. (*On the Equilibrium of Planes*, I. 6, 7)

Similarly, all other straight lines parallel to DE and meeting the arc of the parabola, (1) the portion intercepted between FC, AC with its middle point on KC and (2) a length equal to the intercept between the curve and AC placed with its centre of gravity at H will be in equilibrium about K.

Therefore K is the centre of gravity of the whole system consisting (1) of all the straight lines as MO intercepted between FC, AC and placed as they actually are in the figure and (2) of all the straight lines placed at H equal to the straight lines as PO intercepted between the curve and AC.

And, since the triangle CFA is made up of all the parallel lines like MO, and the segment CBA is made up of all the straight lines like PO within the curve, it follows that the triangle, place where it is in the figure, is in equilibrium about K with the segment CBA placed with its centre of gravity at H.

Divide KC at W so that $CK = 3KW$; then W is the centre of gravity of the triangle ACF; "for this is proved in the books on equilibrium" (Cf. *On the Equilibrium of Planes*, I. 5). Therefore

$$\Delta ACF : (\text{segment } ABC) = HK : KW = 3 : 1.$$

Therefore $\quad \text{segment } ABC = \frac{1}{3}\Delta ACF.$

But $\quad \Delta ACF = 4\Delta ABC.$

Therefore $\quad \text{segment } ABC = \frac{4}{3}\Delta ABC.$

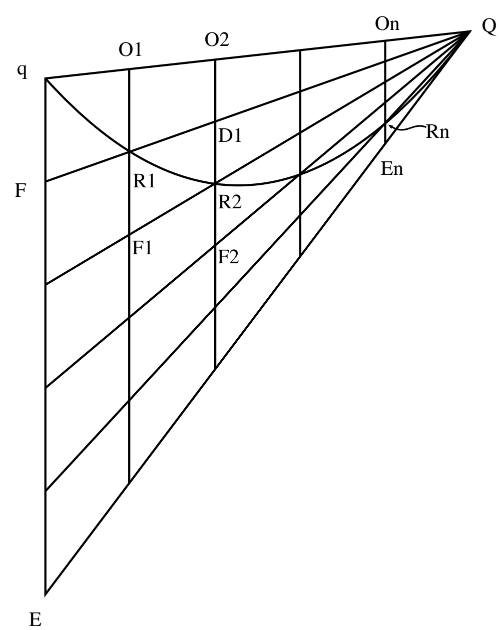
The statement by Archimedes that this is not a proof is found at this point in the *Method*. The mathematically rigorous proof, contained in Propositions 16 and 17 of *Quadrature of the Parabola*, will now be given. Be on the lookout for things like Riemann sums.

Prop. 16. Supposed Qq to be the base of a parabolic segment, q being not more distant than Q from the vertext of the parabola. Draw through q the straight line qE parallel to the axis of the parabola to meet the tangent Q in E. It is required to prove that

$$A(S) = (\text{area of segment}) = \frac{1}{3}\Delta EqQ.$$

The proof will employ the method of exhaustion, a technique much used by Archimedes, and will take the form of a double *reductio ad absurdum*, where the assumptions that the area of the segment is more than and less than $\frac{1}{3}$ the area of the triangle both lead to contradictions.

I. Suppose the area of the segment is greater than $\frac{1}{3}\Delta EqQ$. Then the excess can, if continually added to itself, be made to exceed ΔEqQ. And it is possible to find a submultiple of the triangle EqQ less than the said excess of the segment over $\frac{1}{3}\Delta EqQ$.



Let the triangle FqQ be such a submultiple of the triangle EqQ. Divide Eq into equal parts each equal to qF, and let all points of division including F be joined to Q meeting the parabola in $R_1, R_2, \cdots, R_n$ respectively. Through $R_1, R_2, \cdots, R_n$ draw diameters of the parabola meeting qQ in $O_1, O_2, \cdots, O_n$ respectively. Let O_1R_1 meet QR_2 in F_1, let O_2R_2 meet QR_1 in D_1 and QR_3 in F_2, let O_3R_3 meet QR_2 in D_2 and QR_4 in F_3, and so on.

We have, by hypothesis,

$$\Delta FqQ < A(S) - \frac{1}{3}\Delta EqQ,$$

or,

$$A(S) - \Delta FqQ > \frac{1}{3}\Delta EqQ. \qquad (\alpha)$$

Now, since all the parts of qE, as qF and the rest, are equal, $O_1R_1 = R_1F_1$, $O_2D_1 = R_2F_2$, and so on; therefore

$$\begin{aligned} \Delta FqQ &= (FO_1 + R_1O_2 + D_1O_3 + \cdots) \\ &= (FO_1 + F_1D_1 + \cdots + F_{n-1}D_{n-1} + \Delta E_nR_nQ) \qquad (\beta) \end{aligned}$$

But

$$A(S) < (FO_1 + F_1O_2 + \cdots + F_{n-1}O_n + \Delta E_nO_nQ).$$

Subtracting, we have

$$A(S) - \Delta FqQ < (R_1O_2 + R_2O_3 + \cdots + R_{n-1}O_n + \Delta R_nO_nQ),$$

whence, a *fortiori*, by (α),

$$\frac{1}{3}\Delta EqQ < (R_1O_2 + R_2O_3 + \cdots + R_{n-1}O_n + \Delta R_nO_nQ).$$

But this is impossible, since [Props. 14, 15]

$$\frac{1}{3}\Delta EqQ > (R_1O_2 + R_2O_3 + \cdots + R_{n-1}O_n + \Delta R_nO_nQ).$$

Therefore

$$A(S) > \frac{1}{3}\Delta EqQ$$

cannot be true.

II. If possible, suppose the area of the segment less than $\frac{1}{3}\Delta EqQ$.

Take a submultiple of the triangle EqQ, as the triangle FqQ, less than the excess of $\frac{1}{3}\Delta EqQ$ over the area of the segment, and make the same construction as before.

Since $\Delta FqQ < \frac{1}{3}\Delta EqQ - A(S)$, it follows that

$$\Delta FqQ + A(S) < \frac{1}{3}\Delta EqQ < (FO_1 + \cdots + F_{n-1}O_n + \Delta E_nO_nQ)$$

[Props. 14, 15]. Subtracting from each side the area of the segment, we have

$$\Delta FqQ < (\text{sum of spaces } qFR_1, R_1F_1R_2, \cdots, E_nR_nQ)$$

$$< (FO_1 + F_1D_1 + ... + F_{n-1}D_{n-1} + \Delta E_nR_nQ), \textit{ a fortiori};$$

which is impossible, because, by (β) above,

$$\Delta FqQ = FO_1 + F_1D_1 + \cdots + F_{n-1}D_{n-1} + \Delta E_nR_nQ.$$

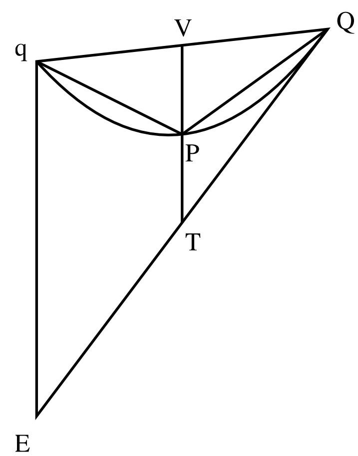
Hence the area of the segment cannot be less than $\frac{1}{3}\Delta EqQ$.

Since the area of the segment is neither less nor greater than $\frac{1}{3}\Delta EqQ$, it is equal to it.

Proposition 17. It is now manifest that the area of any segment of a parabola is four-thirds of the triangle which has the same base as the segment and equal height.

Let Qq be the base of the segment, P its vertex. Then PQq is the inscribed triangle with the same base as the segment and equal height.

Since P is the vertex of the segment, the diameter through P bisects Qq. Let V be the point of bisection. Let VP, and qE drawn parallel to it, meet the tangent at Q in T, E respectively.



Then, by parallels, $qE = 2VT$, and $PV = PT$, [Prop. 2] so that $VT = 2PV$.

Hence $\Delta EqQ = 4\Delta PQq$. But by Prop. 16, the area of the segment is equal to $\frac{1}{3}\Delta EqQ$.

Therefore $$(\text{area of segment}) = \tfrac{4}{3}\Delta PQq.$$

In case readers were not sure what Archimedes meant by the terms *base*, *height*, and *vertex* in the above work, he defines them immediately *after* Prop. 17: "In segments bounded by a straight line and any curve, I call the straight line the base, and the height the greatest perpendicular drawn from the curve to the base of the segment, and the vertex the point from which the greatest perpendicular is drawn." Note that the *vertex of the segment*, as defined by Archimedes, is not necessarily equivalent to the vertex of the parabola.