The General Solution of the Cubic Equation

Cube the third part of the number of "things" to which you add the square of half the
number in the equation, and take the root of the whole, that is, the square root, which
you will use, in the one case adding the half of the number which you just multiplied by
itself, in the other case subtracting the same half, and you will have a "binomial" and
"apotome" respectively; then subtract the cube root of the apotome from the cube root
of the binomial, and the remainder from this is the value of the "thing."

Certainly one of the great theorems of mathematics is the quadratic formula,
but its origins cannot be pinned down to a specific time, place, or person.
The ancient Babylonians of about 2000 B.C. were apparently aware of an
equivalent of the quadratic formula, and the method of completing the square
on which the formula is based. The case of cubic equations is considerably
more difficult, and while some special cases of cubic equations were solved by
the Babylonians and others, including Archimedes, the solution of the general
cubic was not known until much later. In the eleventh century the Persian
poet-mathematician Omar Khayyam devised a geometric solution of cubic
equations, and about 500 years later Italian mathematicians were able to get
an algebraic solution of both cubic and quartic equations. The interested
student can find a discussion of both these methods in *Great Moments in
Mathematics (before 1650)* by Howard Eves. Here, the algebraic solution
will be developed.

The actual solution technique will be applied to cubic equations of the
form $x^3 + mx = n$, or "depressed cubics" which lack a second degree term.
As will be seen presently, any cubic can be converted to this form.

A useful result is the following identity:

$$(a - b)^3 + 3ab(a - b) = a^3 + b^3$$

The student should prove this by expanding and simplifying the left side. If
$x = a - b$, the above identity corresponds to the depressed cubic

$$x^3 + 3abx = a^3 - b^3,$$

which suggests choosing a and b so that

$$m = 3ab \text{ and } n = a^3 - b^3.$$

The solution of these two equations for a and b is

$$a = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} \qquad \text{and} \qquad b = \sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}$$

from which one root, x = a - b, can be found. The cubic now can be factored into a linear factor and a quadratic factor, and the other solutions found by the quadratic formula. The student should verify the above formulas for a and b.

A general cubic of the form $ax^3 + bx^2 + cx + d = 0$ can be "depressed" by means of the substitution $x = z - \frac{b}{3a}$, giving

$$z^3 + \left[-\frac{b^2}{3a^2} + \frac{c}{a}\right] = \left[\frac{2b^3}{27a^3} + \frac{bc}{3a^2} - \frac{d}{a}\right].$$

Some Examples:

1. $3x^3 - 5x^2 - x - 2 = 0$. Here $a = 3$, $b = -5$, $c = -1$, and $d = -2$, so the substitution $x = z + \frac{5}{9}$ yields

$$z^3 - \frac{34}{27}z = \frac{871}{729}$$

where now $m = -\frac{34}{27}$ and $n = \frac{871}{729}$, and so (a calculator is handy for the arithmetic) $z = 1.44444$, to five decimals. Thus $x = 2$, and the equation can be factored as $(x - 2)(3x^2 + x + 1)$, and the other two roots are seen to be complex (but in the 16th century complex numbers were poorly understood).

2. $x^3 - 63x = 162$. This cubic is in depressed form with $m = -63$ and $n = 162$. The numbers a and b turn out to be

$$a = \left[81 + 30\sqrt{-3}\right]^{\frac{1}{3}} \quad \text{and} \quad b = \left[-81 + 30\sqrt{-3}\right]^{\frac{1}{3}}.$$

This presented problems for the 16th century mathematicians, and this situation came to be known as the "irreducible case." The irreducible case will occur whenever the three roots of the cubic are all real and different from zero. In the given example it can be shown that one of the cube roots of $81 + 30\sqrt{-3}$ is $-3 + 2\sqrt{-3}$, and that one of the cube roots of $-81 + 30\sqrt{-3}$ is $3 + 2\sqrt{-3}$. Thus, $a - b = -6$, which is indeed a root of the given cubic.