FAMILIES OF REGULAR SOLUTIONS OF SINGULAR SYSTEMS

L. J. GRIMM[a], B. D. HAILE[b], AND L. M. HALL[a]

[a]Department of Mathematics and Statistics,
University of Missouri-Rolla, Rolla, MO 65409-0020
[b]Department of Mathematics and Statistics, Northwest
Missouri State University, Maryville, MO 64468-6001

ABSTRACT. The seminal 1969 paper of W. A. Harris, Jr., Y. Sibuya, and L. Weinberg provided new proofs for the Perron-Lettenmeyer theorem, as well as several other classical results, and has stimulated renewed consideration of families of regular solutions of certain singular problems. In this paper we give some further applications of the method developed there and, in addition, examine some connections between the Lettenmeyer theorem and an alternative theorem which addresses a problem posed by H. L. Turrittin that dates back to an 1845 example of Briot and Bouquet.

1. INTRODUCTION.

W. A. Harris' mathematical father was H. L. Turrittin. The authors of this paper are Harris' direct descendents (Harris → Grimm → Hall → Haile), and our results illustrate the treatment and development of a particular mathematical theme by five generations of mathematicians. The theme is the study of solutions of singular systems in the neighborhood of a singularity, especially holomorphic solutions. In 1971 Turrittin [31] posed the following problem.

Given the equation

$$\frac{dW}{dz} = \sum_{j=0}^{\infty} z^{-j} A_j \, W + \sum_{j=1}^{\infty} z^{-j} B_j$$

where both series converge if $|z| \geq z_0 > 0$, with a formal solution

$$W(z) = \sum_{j=0}^{\infty} z^{-j} C_j,$$

what are the necessary and sufficient conditions that the formal solution converges?

Special cases of this problem had been solved as long ago as 1845, but in the general form the question was open. Turrittin speculated that the 1969 paper of

Key words and phrases. linear differential equation, regular singular point, holomorphic solution, linear difference equation.

Harris, Y. Sibuya and L. Weinberg [20] might lead to a solution. Indeed, that paper (denoted HSW from now on) has been the genesis of much research, in addition to providing elegant new proofs of a number of classical results, including theorems of O. Perron [29] and F. Lettenmeyer [26]. Grimm and Hall first became interested in the HSW paper in connection with their work on functional differential equations. However, as so often happens in mathematics, one idea led to another, and they found themselves studying singular inhomogeneous problems. This work, although related, did not use the same methods as the HSW paper, and led to a Fredholm alternative type theorem which did settle the question posed by Turrittin. So Turrittin's speculation was both right and wrong. On the one hand, the work which ultimately led to the answer to his question was inspired by HSW, but on the other hand, the actual techniques used were different from those in HSW.

Other results that can, directly or indirectly, be traced back to HSW include the work on difference equations by W. J. Fitzpatrick (another Harris descendent) and Grimm, Hall's work on the exact number of holomorphic solutions, and the recent work on the nth order equation by Haile. First, however, we review some of the classical results concerning holomorphic solutions of singular differential equations and systems.

2. LETTENMEYER'S THEOREM.

The existence of solutions holomorphic at singular points of linear differential equations was shown for a special case of the scalar nth order equation by L. Pochhammer in 1873. The general nth order scalar result was given in 1911 by Perron [29]:

Theorem 1 (Perron). *Let G be a bounded simply connected region containing the origin in the complex plane, and let the scalar functions $a_1(z), \ldots, a_n(z)$ be holomorphic in G. Then if $k < n$, the differential equation*

$$z^k y^{(n)} + a_1(z) y^{(n-1)} + \cdots + a_n(z) y = 0 \tag{1}$$

has at least $n - k$ linearly independent solutions holomorphic at $z = 0$.

The corresponding result for first order systems was obtained in 1926 by Lettenmeyer [26]:

Theorem 2 (Lettenmeyer). *Let $A(z)$ be an $n \times n$ matrix holomorphic at $z = 0$. Let $D = diag(d_1, \ldots, d_n)$ where d_i are nonnegative integers. Then, if $n - \sum_{i=1}^{n} d_i > 0$, the differential system*

$$Ly \equiv z^D \frac{dy}{dz} + A(z) y = 0 \tag{2}$$

has at least $n - \sum_{i=1}^{n} d_i$ linearly independent solutions holomorphic at $z = 0$.

Definition. For any system of the form (2), we will call the integer $\nu = n - \sum_{i=1}^{n} d_i$ the Lettenmeyer estimate for the system.

Later proofs of the theorems of Perron and Lettenmeyer have afforded additional insights into the solution structure. E. Hilb [23] based a new proof of Perron's theorem on the theory of infinite algebraic systems; still another proof was provided in 1956 by M. Iwano [24]. P. Hartman gave a simplified version of Lettenmeyer's proof in his 1964 book [22]. In 1969, Harris, Sibuya and Weinberg provided a new approach to Lettenmeyer's theorem, obtaining it as one of several classical results which are corollaries to the following theorem [20].

Theorem 3 (Harris, Sibuya and Weinberg). *Let $A(z)$ be an $n \times n$ matrix holomorphic at $z = 0$ and let $D = diag(d_1, \ldots, d_n)$, where d_i are nonnegative integers. Then for every positive integer N sufficiently large, and every vector polynomial $\phi(z)$ with $z^D \phi(z)$ of degree N, there exists a polynomial in z of degree $N-1$, $f(z;\phi)$, with coefficients depending on A, N, and ϕ, such that the linear differential system*

$$z^D \frac{dy}{dz} + A(z)y = f(z;\phi) \tag{3}$$

has a solution $y(z)$ holomorphic at $z = 0$. Furthermore, f and y are linear homogeneous functions of the coefficients of ϕ, and

$$z^D(y - \phi) = O(z^{N+1}) \tag{4}$$

as z tends to zero.

The function f is constructed in the proof of the theorem. The system (3) reduces to (2) if

$$f(z;\phi) \equiv 0. \tag{5}$$

This determining equation is in fact a system of Nn linear homogeneous algebraic equations in $Nn + \left(n - \sum_{i=1}^{n} d_i\right)$ unknowns, and thus if $\nu = n - \sum_{i=1}^{n} d_i > 0$, there exist at least ν linearly independent functions ϕ which satisfy (5). However, by the order conditions (4), independent solutions $y(z)$ of (2) correspond to independent ϕ's, and thus Lettenmeyer's theorem holds.

Remark. The HSW theorem itself does not require that $\nu > 0$.

Other proofs of Lettenmeyer's theorem have been given by B. Malgrange [27], J. Mawhin [28], and Ju. F. Korobeĭnik [25] through the use of Fredholm mappings. The results of Korobeĭnik include an extended version of Lettenmeyer's theorem as well as surjectivity conditions on the mapping provided by the differential operator L of equation (2).

3. INHOMOGENEOUS PROBLEMS.

As indicated by the remark in the last section, the HSW theorem actually states that, regardless of how strong the singularity at $z = 0$ is, and regardless of whether the homogeneous equation (2) has a nontrivial holomorphic solution, for every equation of the form (2), there exist infinitely many holomorphic forcing terms $f(z)$ such that the corresponding forced equation

$$z^D \frac{dy}{dz} + A(z)y(z) = f(z) \tag{6}$$

has a solution holomorphic at $z = 0$. This leads to the question: can we characterize the holomorphic functions $f(z)$ for which the system (6) will have a holomorphic solution? Before addressing this question, we apply Lettenmeyer's theorem to several simple scalar examples.

Homogeneous examples. Let a be a complex constant.

(i) $y' + ay = 0$. Here $n = 1$, $d_1 = 0$, and the conclusion of Lettenmeyer's theorem is the standard Picard existence result.

(ii) $zy' + ay = 0$. In this case, each solution is a multiple of z^{-a}. If a is a nonpositive integer, this is holomorphic, otherwise not. Although the Lettenmeyer theorem does not strictly apply, the Lettenmeyer estimate is zero, so in the first instance it is not sharp, but it is sharp in the second.

(iii) $z^2y' + ay = 0$. Every solution is a multiple of $e^{a/z}$ with an essential singularity at the origin if $a \neq 0$. Here, the Lettenmeyer estimate is equal to -1, and there are no holomorphic solutions; in this case the estimate can never be sharp.

Corresponding to these homogeneous cases, we consider related inhomogeneous problems with holomorphic forcing terms.

Inhomogeneous examples. Let $f(z) = \sum_{k=0}^{\infty} f_k z^k$.

(i) The equation $y' + ay = f(z)$ has a holomorphic solution for each holomorphic function $f(z)$.

(ii) We note that for the system $zy' + ay = f(z)$, the solution is $y = cz^{-a} + z^{-a} \int \sum_{k=0}^{\infty} f_k z^{k+a-1} dz$. Thus if $a = -p$, where p is a nonnegative integer, the solution must contain a logarithmic term unless $f_p = 0$. However, if a is not a nonpositive integer, the function $\sum_{k=0}^{\infty} \frac{f_k z^k}{k+a}$ will be a holomorphic solution.

(iii) In 1845, Briot and Bouquet, see [2] p. 75, showed that the equation $z^2y' + ay = f(z)$ has a solution holomorphic at $z = 0$ if, and only if, $\sum_{k=0}^{\infty} \frac{(-1)^k f_{k+1} a^k}{k!} = 0$. In 1899, J. Horn, see [2] p. 76, extended this result to the equation $z^{n+1}y' + a(z)y = f(z)$, where $a(z)$ is holomorphic at $z = 0$.

In his NYU lecture notes [9], K. O. Friedrichs obtained a result for regular singular first-order systems which shows more clearly the alternative nature of the structure:

Theorem 4 (Friedrichs). *Suppose $zA(z)$ and $zf(z)$ are holomorphic at $z = 0$. Then, either the homogeneous system $y' + A(z)y = 0$ has no nontrivial solution holomorphic at $z = 0$, and, in this case, the inhomogeneous system $y'+A(z)y = f(z)$ has exactly one solution holomorphic at zero, or the homogeneous system has at least one nontrivial holomorphic solution. In this case, the inhomogeneous equation has a holomorphic solution if, and only if, the function $f(z)$ satisfies one or several linear conditions.*

For the general, possibly irregular singular, case, the results of Korobeǐnik [25] together with a representation due to A. E. Taylor [30] were utilized to obtain the following alternative theorem [13].

Theorem 5 (Grimm and Hall). *Let $D = diag(d_1, \ldots, d_n)$ and let $A(z)$ be an $n \times n$ matrix holomorphic at $z = 0$. Then the inhomogeneous equation*

$$Ly = z^D \frac{dy}{dz} + A(z)y(z) = f(z) \tag{7}$$

has a solution $y(z)$ holomorphic at $z = 0$ for every function $f(z)$ holomorphic at $z = 0$ if, and only if, the Lettenmeyer estimate for the associated homogeneous equation $Ly = 0$ is sharp, i.e., the homogeneous equation has exactly $n - \sum_{i=1}^{n} d_i$ independent solutions holomorphic at $z = 0$. In the contrary case, (7) will have a solution holomorphic at $z = 0$ if, and only if, the function $f(z)$ is orthogonal to the cokernel $K(L^*)$ of the operator L, where the orthoganility condition is expressible as the limit of a vector Hadamard product.

The orthogonality condition in the space $A_0(D)$ of functions holomorphic in the unit disc D and continuous onto $\overline{D}$ extends the result of Taylor [30] to vector functions. The special features of example (ii) illustrate the following result of Hall [16].

Theorem 6 (Hall). *Let $A(z)$ be an $n \times n$ matrix holomorphic at $z = 0$ and let $A(0)$ have a nonpositive integer eigenvalue. Let $-N$ denote the largest such eigenvalue with m the dimension of the corresponding eigenspace, and let p be a positive integer. Then if $z = 0$ is a regular singular point for $Ly = z^p \frac{dy}{dz} + A(z)y = 0$, there exist m linearly independent polynomials of degree N which belong to $K(L^*)$.*

Remarks.

1. For the operator L above, $\dim K(L^*) = n(p - 1) + \dim K(L)$. Thus, for instance, for the equation $zy' + y = 0$, $K(L) = \{0\}$, so $\dim K(L^*) = 0$, while for the equation $zy' - Ny = 0$, where N is a positive integer, $n = 1$, $p = 1$ once again, but $K(L) = span\{z^N\}$, so $\dim K(L^*) = 1$. In this case there is a polynomial of degree N in $K(L^*)$, but since $\dim K(L^*) = 1$, this polynomial spans $K(L^*)$.
2. For the equation $z^2y'(z)+ay(z) = 0$, the cokernel is spanned by the function $g(z) = ze^{-az}$, i.e.,

$$g(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}a^{k-1}}{(k-1)!} z^k.$$

The Hadamard product in this case is

$$\sum_{k=0}^{\infty} f_k g_k z^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} f_k a^{k-1}}{(k-1)!} z^k$$

and the orthogonality condition is

$$\lim_{\substack{z\to 1 \\ |z|<1}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} f_k\, a^{k-1}}{(k-1)!} z^k = 0,$$

which leads to the result obtained by Briot and Bouquet.

3. After transforming the singularity from ∞ to 0, application of Theorem 5 answers Turrittin's question.

4. THE nTH ORDER CASE.

Here we restrict our considerations to the regular singular point case. let D be the open unit disk in the complex plane, and A_p the Banach space of functions $v(z)$ holomorphic in D, continuous and p times continuously differentiable on $\overline{D}$, with norm

$$\|v\|_p = \max\left\{|v^{(i)}(z)|, 0 \le i \le p, |z| = 1\right\},$$

and define the operator $L : A_p \to A_0$ by

$$Ly(z) = z^n y^{(n)}(z) + z^{n-1} a_1(z) y^{(n-1)}(z) + \cdots + a_n(z) y(z)$$

where $a_i(z) = \sum\limits_{j=0}^{\infty} a_{i,j} z^j \in A_0$.

The indicial equation for the equation $Ly = 0$ is

$$F(r) \equiv r^{(n)} + a_{1,0} r^{(n-1)} + \cdots + a_{n-1,0} r^{(1)} + a_{n,0} = 0,$$

where $r^{(k)}$ is the factorial function $r^{(k)} = r(r-1)\cdots(r-k+1)$. An nth order scalar result [15] corresponding to Theorems 5 and 6 is the following theorem:

Theorem 7 (Haile). *If the equation $F(r) = 0$ has no nonnegative integer root, the cokernel of L is trivial, thus, analogous to Theorems 4 and 5, the equation $Ly = 0$ has a solution in A_n for each $g \in A_0$. If the equation $F(r) = 0$ has exactly one nonnegative integer root, r_1, the cokernel of L is spanned by a single polynomial of degree r_1.*

A general version of Theorem 7, see [15], gives a complete description of the cokernel of the nth order regular singular operator L. The cokernel is spanned by polynomials, and bounds on their degrees are given.

Remark. Recall that, in contrast, in example (iii), with an irregular singularity, the cokernel is spanned by ze^{-az}.

5. FURTHER APPLICATIONS AND EXACT ESTIMATES.

Other corollaries of the HSW theorem include a new proof of the convergence of formally holomorphic solutions in the regular singular case and upper and lower bounds on the number of meromorphic solutions. E. Wagenführer [32] obtained Lettenmeyer estimates for meromorphic solutions as well as extensions of the Lettenmeyer theorem in cases where the Lettenmeyer estimate is negative. Grimm and Hall [12] considered a special case of the latter which also extended to a class of functional differential equations. W. Walter [33] showed that, in the regular singular case, formal logarithmic solutions will actually converge. In [19] Harris gave a simple treatment of the complete convergence proof for all formal solutions in the regular-singular case.

In the general, possibly irregular singular case, the Lettenmeyer theorem gives a lower bound on the dimension of the holomorphic solution subspace. However, the exact dimension of this subspace can be found if the dimension of $K(L^*)$ is known, since the deficiency of the Lettenmeyer estimate is equal to the dimension of $K(L^*)$. Hall [17] developed an algebraic procedure, involving rank considerations, which determines this dimension. In his Strasbourg dissertation (see [10]), P. Gonzalez used the method of Hall, correcting a statement there (replacing a matrix by its transpose), to obtain in addition an exact estimate of the dimension of the meromorphic subspace.

6. NONLINEAR EQUATIONS.

Lettenmeyer's theorem was extended to nonlinear systems by R. W. Bass [1] who obtained the following result.

Theorem 8 (Bass). *Let $D = diag(d_1, \dots, d_n)$ be an $n \times n$ matrix, with d_i non-negative integers. Let $f(z, y)$ be an n-vector function of z and $y = (y_1, \dots, y_n)^T$, holomorphic in some neighborhood of the origin in $(n+1)$-dimensional complex space. Suppose further that $f(z, 0) \equiv 0$ so that f can be written in the form $f(z, y) = A(z)y + g(z, y)$, where $A(z)$ is holomorphic at $z = 0$ and $g(z, y)$ has a convergent expansion in powers of $y_1, \dots, y_n$, all of whose terms are of second or higher degree in the y_j. Then, if $n - \sum_{i=1}^{n} d_i > 0$, the system of differential equations*

$$z^D \frac{dy}{dz} = f(z, y)$$

has at least an $\left(n - \sum_{i=1}^{n} d_i\right)$-parameter algebroid family of solutions holomorphic at $z = 0$.

The HSW method has greatly simplified the proof of this result also, see [18].

The Lettenmeyer and Bass theorems, together with related results, have been extended to various classes of functional differential equations by Fitzpatrick, Grimm and Hall [8], B. L. J. Braaksma and Harris [3], and others, see for instance [11], [12].

7. DIFFERENCE EQUATIONS.

Fitzpatrick and Grimm utilized the HSW method to obtain a version of the Lettenmeyer theorem for difference equations. Let $X(\delta)$ denote the set of all complex-valued functions $g(z)$ with absolutely convergent inverse factorial series

$$g(z) = g_0 + \sum_{k=0}^{\infty} \frac{g_{k+1}k!}{z(z+1)\dots(z+k)} \equiv g_0 + \sum_{k=0}^{\infty} g_{k+1}k!(z-1)^{(-k-1)}$$

for $Re\ z \geq \delta > 1$, and let $X_n(\delta)$ denote the set of n-vector valued functions whose components are elements of $X(\delta)$. The HSW procedure yields the following result [5].

Theorem 9 (Fitzpatrick and Grimm). *Let $A(z)$ and $B(z)$ be $n \times n$ matrices whose elements belong to $X(\delta)$. Let $D = diag(d_1, \dots, d_n)$ with each d_i equal to either 1 or 2. Denote by $(z-1)^{(D)}$ the matrix $\big((z-1)^{(d_1)}, \dots (z-1)^{(d_n)}\big)$. Then if d denotes the number of d_i which are equal to 2, the system*

$$(z-1)^{(D)}\Delta_{-1}y(z) = A(z)y(z) + B(z)y(qz), \tag{8}$$

where $q > 1$ is a constant and Δ_{-1} is the standard backward difference, has at least d linearly independent solutions in $X_n(\delta)$.

Remarks.

1. This is a difference equations analogue of the theorem of Lettenmeyer in the regular singular case only. We do not know of a corresponding result for difference equations with an irregular singularity at infinity.
2. Theorem 9 can also be proved by using Fredholm mappings, see [14].
3. If $d_i = 2$ for all i, then (7) has n linearly independent solutions in $X_n(\delta)$. In this case the solutions of (8) are analogous to solutions of a linear differential system which has $z = \infty$ as an ordinary point.
4. The use of Waring's formula allows the case where some $d_i > 2$ to be reduced to $d_i = 2$ before application of Theorem 9. Thus Theorem 9 can be used in all cases where the d_j's are positive integers. Furthermore, a result of Harris and Turrittin [21] on factorial series representation of reciprocals of factorial series permits the reduction of a system of the form

$$z^{(k)}F(z)\Delta_{-1}y(z) = A(z)y(z) + B(z)y(qz),$$

   where F, A and B have factorial series representations and k is a positive integer, to a system of the type considered here.
5. The HSW procedure also provides a result corresponding to Theorem 8 for nonlinear difference equations [7].

References

1. R. W. Bass, *On the regular solutions at a point of singularity of a system of nonlinear differential equations*, Amer. J. Math. **77** (1955), 734-742.
2. L. Bieberbach, *Theorie der Gewöhnlichen Differentialgleichungen*, 2. Aufl., Springer-Verlag (1965).

3. B. L. J. Braaksma and W. A. Harris, Jr., *Laplace integrals and factorial series in singular functional differential systems*, Applicable Analysis **8** (1978), 23-45.
4. W. J. Fitzpatrick and L. J. Grimm, *A class of singular neutral-differential systems*, Proc. Amer. Math. Soc. **65** (1977), 113-116.
5. W. J. Fitzpatrick and L. J. Grimm, *Convergent factorial series solutions of linear difference systems*, J. Differential Equations **29** (1978), 345-361.
6. W. J. Fitzpatrick and L. J. Grimm, *Linear functional-differential equations in a Banach algebra*, Canad. Math. Bull. **21** (1978), 435-439.
7. W. J. Fitzpatrick and L. J. Grimm, *On solutions of nonlinear difference systems*, Funkcial. Ekvac. **23** (1980), 17-23.
8. W. J. Fitzpatrick, L. J. Grimm and L. M. Hall, *Singular point theory and a class of complex functional differential equations*, Differencial'nye Urvnenija s Otklonjajuscimsja Argumentom, Naukova Dumka, Kiev, 1977, pp. 102-113 (in Russian).
9. K.O. Friedrichs, *Advanced Ordinary Differential Equations*, Gordon and Breach, New York, 1965.
10. P. Gonzalez, *Dimension of the vector space of analytic solutions of certain singular differential equations*, Analysis **10** (1990), 337-372.
11. L. J. Grimm and L. M. Hall, *Holomorphic solutions of functional differential systems near singular points*, Proc. Amer. Math. Soc. **42** (1974), 167-170.
12. L. J. Grimm and L. M. Hall, *Holomorphic solutions of singular functional differential equations*, J. Math. Anal. Appl. **50** (1975), 627-638.
13. L. J. Grimm and L. M. Hall, *An alternative theorem for singular differential systems*, J. Differential Equations **18** (1975), 411-422.
14. L. J. Grimm and W. D. Noble, *Factorial series solutions of linear difference equations*, Differential Equations and Applications, Rousse, 1987, pp. 695-699.
15. B. D. Haile, *Analytic Solutions of nth Order Differential Equations at a Singular Point*, Ph.D. Thesis, University of Missouri-Rolla, Rolla, MO, 1997.
16. L. M. Hall, *A characterization of the cokernel of a singular Fredholm differential operator*, J. Differential Equations **24** (1977), 1-7.
17. L. M. Hall, *The number of analytic solutions of a singular differential system*, J. Differential Equations **32** (1979), 274-284.
18. W. A. Harris, Jr., *Holomorphic solutions of nonlinear differential equations at singular points*, SIAM Stud. Appl. Math. **5** (1969), 184-187.
19. W. A. Harris, Jr., *Logarithmic solutions of linear differential systems with a singularity of the first kind*, Applicable Analysis **8** (1978), 171-174.
20. W. A. Harris, Jr., Y. Sibuya and L. Weinberg, *Holomorphic solutions of linear differential systems at singular points*, Arch. Rational Mech. Anal. **35** (1969), 245-248.
21. W. A. Harris, Jr. and H. L. Turrittin, *Reciprocals of inverse factorial series*, Funkcial. Ekvac. **6** (1964), 37-46.
22. P. Hartman, *Ordinary Differential Equations*, Wiley, New York, 1964.
23. E. Hilb, *Über diejenigen Integrale linearer Differentialgleichungen, welche sich an einer Unbestimmtheitsstelle bestimmt verhalten*, Math. Ann. **82** (1921), 40-41.
24. M. Iwano, *Sur les points singuliers d'une équation différentielle ordinaire linéaire du n-iéme ordre*, J. Fac. Sci. Univ. Tokyo Sect. I **7** (1956), 343-351.
25. Ju. F. Korobeĭnik, *Normal solvability of linear differential equations in the complex plane*, Math. USSR Izvestija **6** (1972), 445-466.
26. F. Lettenmeyer, *Über die an einer Unbestimmtheitsstelle regulären Lösungen eines Systems homogener linearen Differentialgleichungen*, S.-B. Bayer. Akad. Wiss. München Math.-nat. Abt. (1926), 287-307.
27. B. Malgrange, *Remarques sur les points singuliers des équations différentielles*, C. R. Acad. Sci. Paris **273** (1971), Serie A, 1136-1137.
28. J. Mawhin, *Fredholm mappings and solutions of linear differential equations at singular points*, Ann. Mat. Para. Appl. **108** (1976), 329-335.
29. O. Perron, *Über diejenigen Integrale linearer Differentialgleichungen, welche sich an einer Unbestimmtheitsstelle bestimmt verhalten*, Math. Ann. **70** (1911), 1-32.

30. A. E. Taylor, *Banach spaces of functions analytic in the unit circle, I, II*, Studia Math. **11,12** (1950, 1951), 145-170, 25-50..
31. H. L. Turrittin, *My mathematical expectations*, Symposium on Ordinary Differential Equations, Lecture Notes in Mathematics, vol. 312, Springer-Verlag, 1973, pp. 1-22.
32. E. Wagenfuhrer, *Über regulär-singuläre Lösungen von Systemen linearer Differentialgleichungen, I, II*, J. Reine Angew. Math. **267, 272** (1974), 90-114, 150-172.
33. W. Walter, *Gewöhnliche Differentialgleichungen*, Springer-Verlag, Berlin, 1972.