Special Functions

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Chapter 1. Euler, Fourier, Bernoulli, Maclaurin, Stirling

1.1. The Integral Test and Euler's Constant

Suppose we have a series $\sum_{k=1}^{\infty} u_k$ of decreasing terms and a decreasing function f such that $f(k) = u_k$, $k = 1, 2, 3, \ldots$. Also assume f is positive, continuous for $x \geq 1$, and $\lim_{x \to \infty} f(x) = 0$.

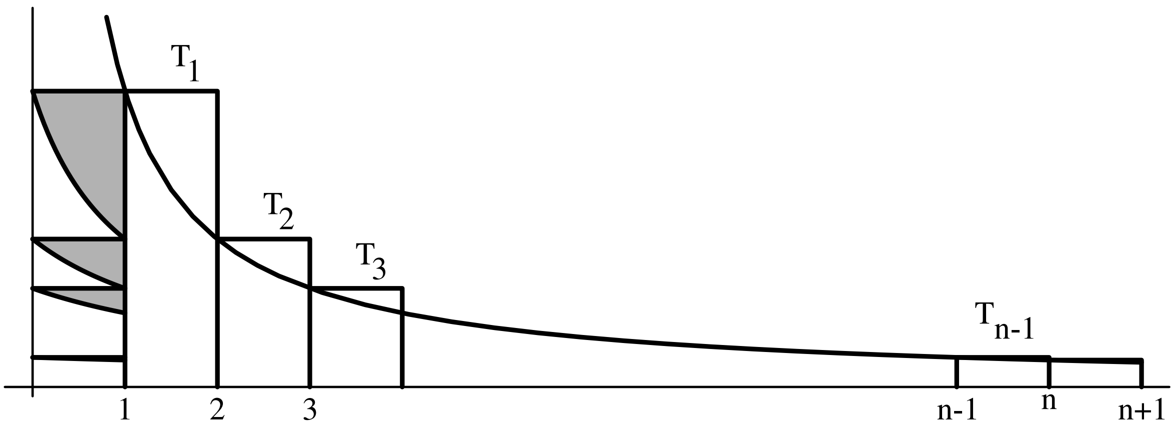

Figure 1

Look at Figure 1 to convince yourself that

$$\sum_{k=1}^{n} u_k = \int_1^n f(x)\,dx + |T_1| + |T_2| + \cdots + |T_{n-1}| + u_n.$$

The left side is the sum of the areas of the rectangles on unit bases with heights $u_1, u_2, \ldots, u_n$ determined from the left end point. $|T_k|$ denotes the area of the triangular-shaped pieces T_k bounded by $x = k+1$, $y = u_k$, and $y = f(x)$. Slide all the $T_k s$ left into the rectangle with opposite vertices $(0,0)$ and $(1, u_1)$ and set

$$A_n = |T_1| + |T_2| + \cdots + |T_{n-1}|$$

Clearly (make sure it *is* clear), $0 < A_2 < A_3 < \cdots < A_n < u_1$, so $\{A_n\}$ is a bounded monotone sequence which has a limit:

$$0 < \lim_{n \to \infty} A_n = \lim_{n \to \infty} [|T_1| + |T_2| + \cdots + |T_{n-1}|] = C \leq u_1.$$

Let $C_n = A_n + u_n$. We have proved the following result, which should be somewhat familiar.

Theorem 1.1.1 (Integral Test). *Let f be positive, continuous and decreasing on $x \geq 1$. If $f(x) \to 0$ as $x \to \infty$, and if $f(k) = u_k$ for each $k = 1, 2, 3, \ldots$, then the sequence of constants $\{C_n\}_{n=1}^{\infty}$ defined by*

$$\sum_{k=1}^{n} u_k = \int_1^n f(x)\,dx + C_n$$

converges, and $0 \leq \lim_{n \to \infty} C_n = C \leq u_1$.

Corollary 1.1.1 (Calculus Integral Test). *Let f be positive, continuous and decreasing on $x \geq 1$. If $f(x) \to 0$ as $x \to \infty$, and if $f(k) = u_k$ for each $k = 1, 2, 3, \ldots$, then the series*

$$\sum_{k=1}^{\infty} u_k$$

converges if and only if the improper integral

$$\int_1^\infty f(x)\,dx$$

converges.

Example 1.1.1 (The Harmonic Series). $f(x) = 1/x$, $u_k = 1/k$. *By the theorem, the sequence* $\{\gamma_n\}$ *defined by*

$$\sum_{k=1}^{n} \frac{1}{k} = \int_1^n \frac{1}{x}\,dx + \gamma_n$$

converges, say to γ, *where*

$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log n \right].$$

The number γ *is called Euler's constant, or the Euler-Mascheroni constant and has value*

$$\gamma = 0.5772\ 15664\ 90153\ 28606\ 06512\ 09008\ldots$$

It is currently not known whether γ *is even rational or not, let alone algebraic or transcendental.*

Exercise 1.1.1. *Use the above definition and Mathematica or Maple to find the smallest value of* n *for which* γ *is correct to four decimal places. Later, we will develop a better way to get accurate approximations of* γ.

Example 1.1.2 (The Riemann Zeta Function). $f(x) = 1/x^s$, $s > 1$. *Now the theorem gives*

$$\sum_{k=1}^{n} \frac{1}{k^s} = \frac{1}{s-1}\left(1 - \frac{1}{n^{s-1}}\right) + C_n(s)$$

where $0 < C_n(s) < 1$. *Let* $n \to \infty$, *giving*

$$\sum_{k=1}^{\infty} \frac{1}{k^s} = \frac{1}{s-1} + C(s)$$

with $0 < C(s) < 1$. *The summation is the real part of the Riemann zeta function,* $\zeta(s)$, *a function with many interesting properties, most of which involve its continuation into the complex plane. However, for the real part we get that*

$$\zeta(s) = \frac{1}{s-1} + C(s),$$

where $0 < C(s) < 1$.

We shall return to both these examples later.

1.2. Fourier Series

Let $L > 0$ and define the functions $\left\{\phi_k(x)\right\}_{k=1}^{\infty}$ on $[0, L]$ by

$$\phi_k(x) = \sqrt{\frac{2}{L}} \sin \frac{k\pi x}{L}.$$

Exercise 1.2.1. *Verify that these functions satisfy*

$$\int_0^L \Big|\phi_k(x)\Big|^2 \, dx = 1,$$

and, if $j \neq k$,

$$\int_0^L \phi_j(x)\, \phi_k(x)\, dx = 0.$$

If these two conditions are satisfied, we call $\{\phi_k(x)\}_{k=1}^{\infty}$ an orthonormal set *over $[0, L]$.*

Now let f be defined on $[0, L]$, and assume that $\int_0^L f(x)\, dx$ and $\int_0^L \Big|f(x)\Big|^2 dx$ both exist. Define the *Fourier coefficients* of f by

$$a_k = \int_0^L f(x)\, \phi_k(x)\, dx.$$

We want to approximate $f(x)$ by a linear combination of a finite subset of the above orthonormal set.

Exercise 1.2.2. *Show that, for any positive integer n,*

$$\int_0^L \Big|f(x) - \sum_{k=1}^{n} c_k \phi_k(x)\Big|^2 dx = \int_0^L \Big|f(x)\Big|^2 dx - \sum_{k=1}^{n} \Big|a_k\Big|^2 + \sum_{k=1}^{n} \Big|c_k - a_k\Big|^2,$$

and that the left side of this expression is a minimum when $c_k = a_k$, $k = 1, 2, \ldots, n$. Note that this is a least squares problem.

So, $\int_0^L \Big|f(x) - \sum_{k=1}^n a_k \phi_k(x)\Big|^2 dx = \int_0^L \Big|f(x)\Big|^2 dx - \sum_{k=1}^n \Big|a_k\Big|^2$, and, since the left side cannot be negative,

$$\sum_{k=1}^{n} \Big|a_k\Big|^2 \leq \int_0^L \Big|f(x)\Big|^2 dx.$$

Since this inequality is true for all n, we have *Bessel's Inequality*:

$$\sum_{k=1}^{\infty} \Big|a_k\Big|^2 \leq \int_0^L \Big|f(x)\Big|^2 dx.$$

Notice that the important thing about the set $\{\phi_k(x)\}$ was that it was an orthonormal set. The specific sine functions were not the main idea. Given an orthonormal set and a function f, we call $\sum_1^\infty a_k\, \phi_k(x)$ the *Fourier series* of f. For our purposes, the most important orthonormal sets are those for which

$$\lim_{n\to\infty} \int_0^L \Big|f(x) - \sum_{k=1}^{n} a_k \phi_k(x)\Big|^2 dx = 0.$$

Orthonormal sets with this property are *complete*. Some examples of complete orthonormal sets follow. The first two are defined on $[0, L]$ and the third one on $[-L, L]$.

$$\left\{ \sqrt{\frac{2}{L}} \sin \frac{k\pi x}{L} \right\}_{k=1}^{\infty} \qquad (ON1)$$

$$\left\{\sqrt{\frac{1}{L}}, \sqrt{\frac{2}{L}}\cos\frac{\pi x}{L}, \sqrt{\frac{2}{L}}\cos\frac{2\pi x}{L}, \dots\right\} \tag{ON2}$$

$$\left\{\sqrt{\frac{1}{2L}}, \sqrt{\frac{1}{L}}\cos\frac{\pi x}{L}, \sqrt{\frac{1}{L}}\sin\frac{\pi x}{L}, \sqrt{\frac{1}{L}}\cos\frac{2\pi x}{L}, \sqrt{\frac{1}{L}}\sin\frac{2\pi x}{L}, \dots\right\} \tag{ON3}$$

There are other complete orthonormal sets, some of which we will see later.

For a given orthonormal set, the Fourier series $\sum_{k=1}^{\infty} a_k\, \phi_k(x)$ is equal to $f(x)$ on $-\infty < x < \infty$ for periodic functions f with period $2L$ provided

(1) f is bounded and piecewise monotone on $[-L, L]$,

(2) $\lim_{h\to 0} \dfrac{f(x+h) + f(x-h)}{2} = f(x)$,

(3) f is odd when $(ON1)$ is the orthonormal set,

(4) f is even when $(ON2)$ is the orthonormal set.

1.3. Bernoulli Functions and Numbers

The *Bernoulli functions*, $B_0(x), B_1(x), B_2(x), \dots$, satisfy the following conditions on $-\infty < x < \infty$:

$$B_0(x) = 1$$

$$B_n'(x) = B_{n-1}(x),\ \ n = 1, 2, 3, \dots *$$

$$\int_0^1 B_n(x)\, dx = 0,\ \ n = 1, 2, 3, \dots$$

$$B_n(x+1) = B_n(x),\ \ n = 1, 2, 3, \dots$$

Exercise 1.3.1. *Show that there exist constants* $B_0, B_1, B_2, \dots$ *such that for* $0 < x < 1$

$$B_0(x) = \frac{B_0}{0!0!}$$

$$B_1(x) = \frac{B_0 x}{0!1!} + \frac{B_1}{1!0!}$$

$$B_2(x) = \frac{B_0 x^2}{0!2!} + \frac{B_1 x}{1!1!} + \frac{B_2}{2!0!}$$

$$B_3(x) = \frac{B_0 x^3}{0!3!} + \frac{B_1 x^2}{1!2!} + \frac{B_2 x}{2!1!} + \frac{B_3}{3!0!}$$

etc.

Exercise 1.3.2. *Show that , when* $n \geq 2$, $B_n = n!\, B_n(0)$

Exercise 1.3.3. *Show that on* $(0, 1)$,

$$0! B_0(x) = B_0$$

* Except when $n = 1$ or 2 and x is an integer.

$$1!B_1(x) = B_0x + B_1$$
$$2!B_2(x) = B_0x^2 + 2B_1x + B_2$$
$$3!B_3(x) = B_0x^3 + 3B_1x^2 + 3B_2x + B_3$$
$$etc.$$

Some authors define the Bernoulli polynomials (on $(-\infty, \infty)$) to be the right hand sides of the above equations. If, in the future, you encounter Bernoulli functions or polynomials, be sure to check what is intended by a particular author.

Exercise 1.3.4. *Show that for $n \geq 2$, $B_n(1) = B_n(0)$.*

Exercise 1.3.5. *Compute B_n for $n = 0, 1, 2, 3, \ldots, 12$.*

Exercise 1.3.6. *Show that $B_1(x) = x - \lfloor x \rfloor - 1/2$ for $-\infty < x < \infty$ and x not an integer. [Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.]*

Since $B_1(x) = x - \frac{1}{2}$ on $(0, 1)$ and is an odd function on $(-1, 1)$ (do you see why?) we can expand it in Fourier series using $(ON1)$ with $L = 1$. The Fourier coefficients are

$$a_k = \sqrt{2} \int_0^1 (x - \frac{1}{2}) \sin(k\pi x)\, dx = -\frac{\sqrt{2}}{k\pi} \left(\frac{1 + (-1)^k}{2} \right).$$

Thus, $a_k = 0$ if k is odd, and $a_k = -\frac{\sqrt{2}}{k\pi}$ if k is even. This gives

$$B_1(x) = -2 \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{2k\pi} = -\frac{2}{2\pi} \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k}.$$

Integrate term by term and use the fact that $B_2'(x) = B_1(x)$ to get

$$B_2(x) = \frac{2}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^2}.$$

Similarly,

$$B_{2n+1}(x) = (-1)^{n+1} \frac{2}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^{2n+1}},$$

and

$$B_{2n}(x) = (-1)^{n+1} \frac{2}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^{2n}}.$$

Exercise 1.3.7. *The work above with the Fourier series was done formally, without worrying about whether the results were meaningful. Prove that the formulas for $B_2(x)$, $B_{2n+1}(x)$, and $B_{2n}(x)$ are correct by showing that the series converge and satisfy the properties of the Bernoulli functions.*

Exercise 1.3.8. *Use Mathematica or Maple to plot graphs of $B_1(x)$, $B_2(x)$, and $B_3(x)$ on $0 \leq x \leq 4$. Also graph the Fourier approximations of $B_1(x)$, $B_2(x)$, and $B_3(x)$ using $n = 2$, $n = 5$, and $n = 50$.*

Example 1.3.1 (Some Values of the Riemann Zeta Function). *Since $B_n(0) = B_n/n!$, we have $B_2(0) = 1/12$. Therefore,*

$$\frac{1}{12} = \frac{2}{(2\pi)^2} \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \right)$$

and so we get

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{(2\pi)^2}{(12)(2)} = \frac{\pi^2}{6}.$$

Exercise 1.3.9. *Find* $\zeta(4)$, $\zeta(6)$, *and* $\zeta(8)$.

Van der Pol used to say that those who know these formulas are mathematicians and those who do not are not.

1.4. The Euler-Maclaurin Formulas

Let p and q be integers and assume f is differentiable (as many times as needed) for $p \leq x \leq q$. Let k be an integer, $p \leq k < q$. Then

$$\int_k^{k+1} f(x)\,dx = \int_k^{k+1} f(x)B_0(x)\,dx = \lim_{\epsilon\to 0}\int_{k+\epsilon}^{k+1-\epsilon} f(x)B_1'(x)\,dx.$$

Integration by parts gives

$$\int_k^{k+1} f(x)\,dx = \lim_{\epsilon\to 0}\left[f(x)B_1(x)\Big]_{k+\epsilon}^{k+1-\epsilon} - \int_{k+\epsilon}^{k+1-\epsilon} f'(x)B_1(x)\,dx\right] = \frac{f(k)+f(k+1)}{2} - \int_k^{k+1} f'(x)B_1(x)\,dx.$$

Adding between p and q, we get

$$\int_p^q f(x)\,dx = \sum_{k=p}^{q-1}\int_k^{k+1} f(x)\,dx = \sum_{k=p}^{q} f(k) - \frac{f(p)+f(q)}{2} - \int_p^q f'(x)B_1(x)\,dx.$$

A slight rearrangement produces the first *Euler-Maclaurin Formula*:

$$\sum_{k=p}^{q} f(k) = \int_p^q f(x)\,dx + \frac{f(p)+f(q)}{2} + \int_p^q f'(x)B_1(x)\,dx. \tag{EM1}$$

This is a useful formula for estimating sums.

Additional Euler-Maclaurin formulas can be obtained by further integration by parts.

Exercise 1.4.1. *Derive the following: (Remember that* $B_j = 0$ *if* $j \geq 3$ *and odd.)*

$$\sum_{k=p}^{q} f(k) = \int_p^q f(x)\,dx + \frac{f(p)+f(q)}{2} + \frac{f'(q)-f'(p)}{12} - \int_p^q f''(x)B_2(x)\,dx. \tag{EM2}$$

$$\sum_{k=p}^{q} f(k) = \int_p^q f(x)\,dx + \frac{f(p)+f(q)}{2} + \frac{f'(q)-f'(p)}{12} + \int_p^q f'''(x)B_3(x)\,dx. \tag{EM3}$$

$$\sum_{k=p}^{q} f(k) = \int_p^q f(x)\,dx + \frac{f(p)+f(q)}{2} + \sum_{j=2}^{m}\left(f^{(j-1)}(q) - f^{(j-1)}(p)\right)\frac{B_j}{j!} + (-1)^{m+1}\int_p^q f^{(m)}(x)B_m(x)\,dx. \tag{EMm}$$

Example 1.4.1. *In (EM3), let $f(x) = x^2$, $p = 0$, and $q = n$. Since $f^m(x) = 0$ for $m \geq 3$ we get*

$$\begin{aligned}\sum_{k=0}^{n} k^2 &= \int_0^n x^2\,dx + \frac{0+n^2}{2} + \frac{2n-0}{12} \\ &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \\ &= \frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

This is much neater than mathematical induction.

Example 1.4.2. *In (EMm), let $p = 0$, $q = n$, $m = s$, and $f(x) = x^s$, where s is a positive integer other than 1. Then*

$$\begin{aligned}\sum_{k=0}^{n} k^s &= \frac{n^{s+1}}{s+1} + \frac{n^s}{2} + \sum_{j=2}^{s} \frac{f^{(j-1)}(n)B_j}{j!} + (-1)^{s+1}\int_0^n s!B_s(x)\,dx \\ &= \frac{n^{s+1}}{s+1} + \frac{n^s}{2} + \sum_{j=2}^{s} \frac{s(s-1)\dots(s-j+2)n^{s-j+1}B_j}{j!} \\ &= n^s + \frac{1}{s+1}\sum_{j=0}^{s} \binom{s+1}{j} n^{s+1-j}B_j\end{aligned}$$

Exercise 1.4.2. *Fill in the details in the last example and get formulas for $\sum_{k=1}^{n} k^3$ and $\sum_{k=1}^{n} k^4$.*

In some cases, as $x \to \infty$, $f^{(m)}(x) \to 0$ for m large enough. When the integral in the following expression converges, we can define a constant C_p by

$$C_p = \frac{f(p)}{2} - \sum_{j=2}^{m} \frac{f^{(j-1)}(p)B_j}{j!} + (-1)^{m+1}\int_p^\infty f^{(m)}(x)B_m(x)\,dx.$$

Exercise 1.4.3. *Show that C_p is independent of m by showing that the right side is unchanged when m is replaced by $m+1$. Integration by parts helps.*

Subtract the C_p equation from (EMm) to get

$$\sum_{k=p}^{q} f(k) = C_p + \int_p^q f(x)\,dx + \frac{f(q)}{2} + \sum_{j=2}^{m} \frac{f^{(j-1)}(q)B_j}{j!} + (-1)^m \int_q^\infty f^{(m)}(x)B_m(x)\,dx.$$

We solve for C_p to get

$$C_p = \sum_{k=p}^{q} f(k) - \int_p^q f(x)\,dx - \frac{f(q)}{2} - \sum_{j=2}^{m} \frac{f^{(j-1)}(q)B_j}{j!} - (-1)^m \int_q^\infty f^{(m)}(x)B_m(x)\,dx.$$

Example 1.4.3 (Euler's Constant). *Let $f(x) = 1/x$, $p = 1$, $q = n$, and (at first) $m = 3$. Then the penultimate formula involving C_p, now C_1, gives*

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k} &= C_1 + \int_1^n \frac{1}{x}\,dx + \frac{1}{2n} + \sum_{j=2}^{3} \frac{f^{(j-1)}(n)B_j}{j!} + (-1)^3\int_n^\infty f^{(3)}(x)B_3(x)\,dx \\ &= \log n + C_1 + \frac{1}{2n} - \frac{B_2}{n^2 2!} - \int_n^\infty -6x^{-4}B_3(x)\,dx \\ &= \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + 6\int_n^\infty \frac{B_3(x)}{x^4}\,dx.\end{aligned}$$

Exercise 1.4.4. *Fill in the details to this point in the example, especially why γ can replace C_1. Then, assuming γ is known, obtain bounds on the last integral and approximate $\sum_{k=1}^{n} \frac{1}{k}$ for $n = 10, 50,$ and 100. How close are your estimates?*

(Continuation of Example 1.4.3). *If now $q = 10$, and m is arbitrary, the last formula for C_1 $(= \gamma)$ gives*

$$\gamma = \sum_{k=1}^{10} \frac{1}{k} - \log 10 - \frac{1}{20} - \sum_{j=2}^{m} \frac{(-1)^{j-1}(j-1)!B_j}{10^j j!} - (-1)^m \int_{10}^{\infty} \frac{(-1)^m m! B_m(x)}{x^{m+1}}\, dx$$
$$= \sum_{k=1}^{10} \frac{1}{k} - \log 10 - \frac{1}{20} + \sum_{j=2}^{m} \frac{B_j}{10^j j} - \int_{10}^{\infty} \frac{m! B_m(x)}{x^{m+1}}\, dx.$$

Exercise 1.4.5. *Prove that if $m = 10$ in the last formula for γ, then the integral is less than 10^{-12}, and so the other terms can be used to compute γ correct to at least ten decimal places. Do this computation. To best appreciate the formula, do the computation by hand, assuming that you know $\log 10$ to a sufficient number of places (you have already found exact values for the Bernoulli numbers you need). ($\log 10 =$ 2.3025 85092 994...)*

1.5 The Stirling Formulas

This section is a (long) derivation of the Stirling formulas for $\log(z!)$ and $z!$. As you work through the section, think about how the steps fit together.

Exercise 1.5.1. *Let $p = 1$, $q = n$, $m \geq 2$, and $f(x) = \log(z + x)$ for $z > -1$. Use (EMm) to get*

$$\sum_{k=1}^{n} \log(z+k) = (z + n + \frac{1}{2}) \log(z+n) - (z + \frac{1}{2}) \log(z+1) - n + 1$$
$$+ \sum_{j=2}^{m} \frac{B_j}{j(j-1)} \left(\frac{1}{(z+n)^{j-1}} - \frac{1}{(z+1)^{j-1}} \right) \qquad (1.5.1)$$
$$+ \int_{1}^{n} \frac{(m-1)! B_m(x)}{(z+x)^m}\, dx.$$

Put $z = 0$ in (1.5.1) to get

$$\log(n!) = (n + \frac{1}{2}) \log n - n + 1 + \sum_{j=2}^{m} \frac{B_j}{j(j-1)} \left(\frac{1}{n^{j-1}} - 1 \right) + \int_{1}^{n} \frac{(m-1)! B_m(x)}{x^m}\, dx. \qquad (1.5.2)$$

In the next chapter we will see how *Wallis' formulas*, (see also A&S, 6.1.49)

$$\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{(2n)!}{2^{2n}(n!)^2} \frac{\pi}{2},$$
$$\int_0^{\pi/2} \sin^{2n+1} x\, dx = \frac{2^{2n}(n!)^2}{(2n)!} \frac{1}{2n+1},$$

can be used to prove that

$$\lim_{n\to\infty} \frac{(2n)! \sqrt{n\pi}}{2^{2n}(n!)^2} = 1. \qquad (1.5.3)$$

Accept this result for now - you will have a chance to prove it later! From (1.5.3) we get

$$\lim_{n\to\infty}\left[\log\left((2n)!\right)+\log\sqrt{n\pi}-2n\log 2-2\log\left(n!\right)\right]=0. \tag{1.5.4}$$

Substitute for $\log((2n)!)$ and $\log(n!)$ in (1.5.4) using (1.5.2) and simplify to get

$$\begin{aligned}\lim_{n\to\infty}\Bigg[&\frac{1}{2}\log 2-1+\frac{1}{2}\log\pi\\ &+\sum_{j=2}^{m}\frac{B_j}{j(j-1)}\left(\frac{1}{(2n)^{j-1}}-\frac{2}{n^{j-1}}+1\right)\\ &+\int_1^{2n}\frac{(m-1)!B_m(x)}{x^m}\,dx-2\int_1^{n}\frac{(m-1)!B_m(x)}{x^m}\,dx\Bigg]=0.\end{aligned}$$

More simplification yields

$$\log\sqrt{2\pi}-1+\sum_{j=2}^{m}\frac{B_j}{j(j-1)}-\int_1^{\infty}\frac{(m-1)!B_m(x)}{x^m}\,dx=0 \tag{1.5.5}$$

Exercise 1.5.2. *Show that*

$$\int_1^{n}\frac{(m-1)!B_m(x)}{x^m}\,dx-\int_1^{\infty}\frac{(m-1)!B_m(x)}{x^m}\,dx=-\int_0^{\infty}\frac{(m-1)!B_m(x)}{(n+x)^m}\,dx \tag{1.5.6}$$

Add (1.5.5) to (1.5.2), and use (1.5.6) to get

$$\log\left(n!\right)=\log\sqrt{2\pi}+(n+\frac{1}{2})\log n-n+\sum_{j=2}^{m}\frac{B_j}{j(j-1)n^{j-1}}-\int_0^{\infty}\frac{(m-1)!B_m(x)}{(n+x)^m}\,dx. \tag{1.5.7}$$

Clearly, for integers $z>0$,

$$\begin{aligned}z!&=\lim_{n\to\infty}1\cdot 2\cdot 3\cdot\ldots\cdot z\\ &=\lim_{n\to\infty}\frac{1\cdot 2\cdot 3\cdot\ldots\cdot z(z+1)(\ldots(z+n)}{(z+1)\ldots(z+n)}\\ &=\lim_{n\to\infty}\left[\left(\frac{n!\,n^z}{(z+1)\ldots(z+n)}\right)\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\ldots\left(\frac{n+z}{n}\right)\right].\end{aligned}$$

Since each of the last factors has limit one, we have (see A&S, 6.1.2), for $z>-1$,

$$z!=\lim_{n\to\infty}\frac{n!\,n^z}{(z+1)(z+2)\ldots(z+n)}. \tag{1.5.8}$$

Taking logs,

$$\log\left(z!\right)=\lim_{n\to\infty}\left[\log\left(n!\right)+z\log n-\sum_{k=1}^{n}\log\left(z+k\right)\right]. \tag{1.5.9}$$

Substitute from (1.5.7) and (1.5.1) to get

$$\begin{aligned}\log\left(z!\right)=\log\sqrt{2\pi}&+(z+\frac{1}{2})\log\left(z+1\right)-(z+1)\\ &+\sum_{j=2}^{m}\frac{B_j}{j(j-1)(z+1)^{j-1}}-\int_1^{\infty}\frac{(m-1)!B_m(x)}{(z+x)^m}\,dx.\end{aligned} \tag{1.5.10}$$

Exercise 1.5.3. *Show that*

$$\lim_{n\to\infty}\left[\left(z+n+\frac{1}{2}\right)(\log(z+n)-\log n)\right]=z \tag{1.5.11}$$

and use this fact to get (1.5.10).

If $z>0$, add $\log(z+1)$ to both sides of (1.5.10)

$$\begin{aligned}\log((z+1)!) &= \log\sqrt{2\pi}+(z+\frac{3}{2})\log(z+1)-(z+1)\\ &+\sum_{j=2}^{m}\frac{B_j}{j(j-1)(z+1)^{j-1}}-\int_1^\infty\frac{(m-1)!B_m(x)}{(z+x)^m}\,dx.\end{aligned}$$

Finally, replace $z+1$ by z:

$$\begin{aligned}\log(z!) &= \log\sqrt{2\pi}+(z+\frac{1}{2})\log z-z\\ &+\sum_{j=2}^{m}\frac{B_j}{j(j-1)z^{j-1}}-\int_0^\infty\frac{(m-1)!B_m(x)}{(z+x)^m}\,dx.\end{aligned} \tag{1.5.12}$$

Note that for $z=n$, (1.5.12) is identical to (1.5.7).

For $z\in\mathbb{C}-\{z\mid\Re(z)\le 0\}$, everything on the right side of (1.5.12) is analytic. Analytic continuation then makes (1.5.12) valid for all complex z not on the non-positive real axis. To make the notation more compact, let

$$E(z)=\sum_{j=2}^{m}\frac{B_j}{j(j-1)z^{j-1}}-\int_0^\infty\frac{(m-1)!B_m(x)}{(z+x)^m}\,dx, \tag{1.5.13}$$

so that (1.5.12) becomes

$$\log(z!)=\log\sqrt{2\pi}+(z+\frac{1}{2})\log z-z+E(z), \tag{1.5.14}$$

or, equivalently,

$$z!=\sqrt{2\pi z}\;z^z e^{-z}e^{E(z)}. \tag{1.5.15}$$

Equations (1.5.14) and (1.5.15) are the *Stirling formulas* for $\log(z!)$ and $z!$. Equation (1.5.15) can be thought of as *defining* $z!$ when z is not a positive integer. See A&S, 6.1.37 and 6.1.38. The term $E(z)$ is small and can be bounded by simple functions, so the Stirling formulas can be used to estimate $z!$ and $\log(z!)$ quite accurately.

Exercise 1.5.4. *For z real and positive, show that*

$$0<E(z)<\frac{1}{12z},$$

and

$$\frac{1}{12z}-\frac{1}{360z^3}<E(z)<\frac{1}{12z}-\frac{1}{360z^3}+\frac{1}{1260z^5}.$$

Exercise 1.5.5. *Use the Stirling formulas to estimate* $5!$ *and* $\log(5!)$ *within 3 decimal places, then do the same for* $5.5!$ *and* $\log(5.5!)$*. Think about how you could find these values without a fancy calculator or computer.*