Special Functions

Leon M. Hall
Professor of Mathematics
University of Missouri-Rolla

Chapter 2. The Gamma Function

2.1. Definition and Basic Properties

Although we will be most interested in real arguments for the gamma function, the definition is valid for complex arguments. See Chapter 6 in A&S for more about the gamma function.

Definition 2.1.1. *For z a complex number with $\Re(z) > 0$, $\Gamma(z) = \displaystyle\int_0^\infty e^{-t} t^{z-1}\, dt$.*

Theorem 2.1.1 (Difference Equation). $\Gamma(z+1) = z\,\Gamma(z)$.

Proof. For the proof we apply integration by parts to the integral in the definition of $\Gamma(z)$.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt = \frac{t^z e^{-t}}{z}\Big]_0^\infty + \int_0^\infty \frac{t^z e^{-t}}{z}\, dt.$$

Thus, $z\,\Gamma(z) = \int_0^\infty e^{-t} t^z\, dt = \Gamma(z+1)$. ♠

Theorem 2.1.2 (Factorial Equivalence). $\Gamma(n+1) = n!$ *for* $n = 0, 1, 2, \ldots$.

Proof. By direct calculation in the definition, $\Gamma(1) = 1$. Repeated use of Theorem 2.1.1 gives $\Gamma(n+1) = n!\,\Gamma(1) = n!$. ♠

Theorem 2.1.3. *If x is real and positive, then* $\displaystyle\lim_{x\to 0+} \Gamma(x) = +\infty$.

Proof.

$$\Gamma(x) > \int_0^1 e^{-t} t^{x-1}\, dt > \frac{1}{e}\int_0^1 t^{x-1}\, dt.$$

The last integral is an improper integral, so

$$\int_0^1 t^{x-1}\, dt = \lim_{\epsilon\to 0+} \int_\epsilon^1 t^{x-1}\, dt = \lim_{\epsilon\to 0+} \left[\frac{1}{x} - \frac{\epsilon^x}{x}\right] = \frac{1}{x}.$$

So, $\Gamma(x) > \dfrac{1}{e\,x}$ for $x > 0$, and thus $\Gamma(x) \to \infty$ as $x \to 0$. ♠

The gamma function is often referred to as the "continuous version of the factorial," or words to that effect. If we are going to say this, we need to prove that $\Gamma(x)$ is continuous. The next theorem uses the Weierstrass M-test for improper integrals, something you should be familiar with for series. The result works similarly for integrals. (Find your advanced calculus book and review the Weierstrass M-test if necessary.)

Theorem 2.1.4 (Continuity of Γ). *The gamma function is continuous for all real positive x.*

Proof. Assume $x_0 > 0$ and choose a and b such that $0 < a < x_0 < b$. Then the integral $\int_1^\infty e^{-t} t^{x-1}\, dt$ converges uniformly on $[a, b]$ by the Weierstrass M-test because $\left|e^{-t} t^{x-1}\right| < e^{-t} t^{b-1}$ and $\int_1^\infty e^{-t} t^{b-1}\, dt$ converges.

The integral $\int_0^1 e^{-t} t^{x-1}\, dt$ is proper for $x \in [a, b]$ if $a \geq 1$. If $0 < a < 1$, then this integral also converges uniformly by the Weierstrass M-test since $\left|e^{-t} t^{x-1}\right| < t^{a-1}$ and $\int_0^1 t^{a-1}\, dt$ converges.

Combining these results, we see that the integral defining $\Gamma(x)$ converges uniformly on $[a, b]$, and the integrand is continuous in x and t. By an advanced calculus theorem, this makes Γ continuous on $[a, b]$ and thus continuous at x_0. ♠

Exercise 2.1.1. *Show that, for x real and positive,* $\lim_{x \to 0+} x\,\Gamma(x) = 1$.

The domain of $\Gamma(x)$ can be extended to include values between consecutive negative integers. For $n = 1, 2, 3, \ldots$, and $-n < x < -n + 1$, define $\Gamma(x)$ by

$$\Gamma(x) = \frac{\Gamma(x+n)}{x(x+1)(x+2)\cdots(x+n-1)}.$$

In this way, $\Gamma(x)$ is defined for all $x \neq 0, -1, -2, \ldots$.

Exercise 2.1.2. *Show that* $\Gamma(x+1) = x\,\Gamma(x)$ *for all* $x \neq 0, -1, -2, \ldots$.

We know that $\Gamma(x)$ becomes infinite as $x \to 0+$ and as $x \to \infty$, but what happens in between? Differentiating, we get, for $0 < x < \infty$,

$$\Gamma'(x) = \frac{d}{dx}\int_0^\infty e^{-t}t^{x-1}\,dt = \int_0^\infty e^{-t}t^{x-1}\log t\,dt,$$

$$\Gamma''(x) = \int_0^\infty e^{-t}t^{x-1}(\log t)^2\,dt.$$

Since the integrand in $\Gamma''(x)$ is positive for $0 < x < \infty$, so is $\Gamma''(x)$. Thus, the graph of $\Gamma(x)$ is concave up on $(0, \infty)$.

The technique used in the proof of the following theorem is one everyone should know.

Theorem 2.1.5. $\displaystyle\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$

Proof. Consider the following regions in the first quadrant of the plane, shown in Figure 2.

$$S = \{(x, y) \,|\, 0 \leq x \leq R,\ 0 \leq y \leq R\}$$

$$D_1 = \{(x, y) \,|\, x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq R^2\}$$

$$D_2 = \{(x, y) \,|\, x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 2R^2\}$$

Clearly,

$$\iint_{D_1} e^{-x^2-y^2}dA < \iint_S e^{-x^2-y^2}dA < \iint_{D_2} e^{-x^2-y^2}dA.$$

Use polar coordinates on the outside integrals and rectangular coordinates on the middle one to get

$$\int_0^{\pi/2}\int_0^R r\,e^{-r^2}dr\,d\theta < \int_0^R\int_0^R e^{-x^2}e^{-y^2}dx\,dy < \int_0^{\pi/2}\int_0^{\sqrt{2}R} r\,e^{-r^2}dr\,d\theta,$$

$$\int_0^{\pi/2}\frac{1-e^{-R^2}}{2}\,d\theta < \left(\int_0^R e^{-x^2}dx\right)\left(\int_0^R e^{-y^2}dy\right) < \int_0^{\pi/2}\frac{1-e^{-2R^2}}{2}\,d\theta,$$

$$\frac{\pi}{4}(1-e^{-R^2}) < \left(\int_0^R e^{-x^2}dx\right)^2 < \frac{\pi}{4}(1-e^{-2R^2}).$$

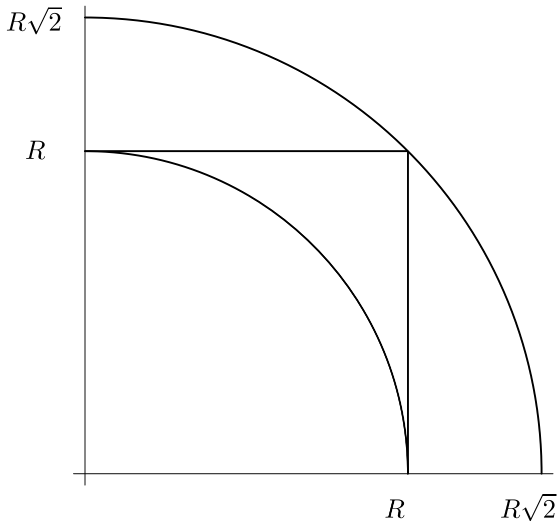


Figure 2.1.1

Taking limits as $R \to \infty$ yields

$$\left(\int_0^\infty e^{-x^2} dx\right)^2 = \frac{\pi}{4},$$

and so $\displaystyle\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. ♠

Corollary 2.1.5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Proof. $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{-1/2} dt$. Let $t = y^2$ to get

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-y^2} y^{-1} 2y\, dy = 2\int_0^\infty e^{-y^2} dy = \sqrt{\pi}. \spadesuit$$

Exercise 2.1.3. *Prove* $\Gamma(n + \frac{1}{2}) = \frac{(2n)!\,\sqrt{\pi}}{2^{2n} n!}$.

Exercise 2.1.4. *For* $0 < x < \infty$, *prove* $\Gamma(x) = 2\int_0^\infty e^{-t^2} t^{2x-1} dt$.

Exercise 2.1.5. *Show that* $f(x) = \int_0^\infty e^{-t^2} \cos(xt)\, dt = \frac{\sqrt{\pi}}{2} e^{-x^2/4}$. *[Hint: Find and solve a differential equation satisfied by* f*.]*

Exercise 2.1.6. *Find all positive numbers* T *such that* $\int_0^T x^{-\log x}\, dx = \int_T^\infty x^{-\log x}\, dx$, *and evaluate the integrals.*

2.2. The Beta Function, Wallis' Product

Another special function defined by an improper integral and related to the gamma function is the *beta function*, denoted $B(x, y)$.

Definition 2.2.1. $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, *for* $x > 0, \quad y > 0$.

If both $x > 1$ and $y > 1$, then the beta function is given by a proper integral and convergence is not a question. However, if $0 < x < 1$ or $0 < y < 1$, then the integral is improper. Convince yourself that in these cases the integral converges, making the beta function well-defined. We now develop some of the properties of $B(x,y)$. Unless otherwise stated, we assume x and y are in the first quadrant.

Theorem 2.2.1 (Symmetry). $B(x,y) = B(y,x)$.

Proof. In the definition, make the change of variable $u = 1-t$. ♠

Theorem 2.2.2. $B(x,y) = 2\displaystyle\int_0^{\pi/2} (\sin t)^{2x-1}(\cos t)^{2y-1}dt$.

Proof. Make the change of variable $t = \sin^2 u$. ♠

Theorem 2.2.3. $B(x,y) = \displaystyle\int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}}\, dt$.

Proof. Let $t = \frac{u}{1+u}$. ♠

Exercise 2.2.1. *Fill in the details in the proofs of Theorems 2.2.1 - 2.2.3.*

Theorem 2.2.4 (Relation to the Gamma Function). $B(x,y) = \dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$.

Proof. The proof uses the method employed in the proof of Theorem 2.1.5. (When a trick is used twice it becomes a method!) From Exercise 2.1.4, we know $\Gamma(x) = 2\displaystyle\int_0^{\infty} e^{-t^2}t^{2x-1}dt$, so consider the function given by $G(t,u) = t^{2x-1}u^{2y-1}e^{-t^2-u^2}$. Integrate G (with respect to t and u) over the three regions shown in Figure 2, using polar coordinates in the quarter-circles as before. The inequalities become

$$\int_0^{\pi/2} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\, d\theta \int_0^R r^{2x+2y-1}e^{-r^2}dr$$
$$< \int_0^R t^{2x-1}e^{-t^2}dt \int_0^R u^{2y-1}e^{-u^2}du$$
$$< \int_0^{\pi/2} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\, d\theta \int_0^{\sqrt{2}R} r^{2x+2y-1}e^{-r^2}dr.$$

As $R \to \infty$, we see from Exercise 2.1.4 and Theorem 2.2.2 that the center term approaches $\Gamma(x)\,\Gamma(y)/4$, and the outside terms approach $B(x,y)\,\Gamma(x+y)/4$. Thus, $B(x,y) = \dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$. ♠

Corollary 2.2.4. $B(\frac{1}{2},\frac{1}{2}) = \pi$.

Exercise 2.2.2 (Dirichlet Integrals 1). *Show that*

$$\iiint_V x^{\alpha-1}y^{\beta-1}z^{\gamma-1}dV = \frac{\Gamma(\frac{\alpha}{2})\,\Gamma(\frac{\beta}{2})\,\Gamma(\frac{\gamma}{2})}{8\,\Gamma(\frac{\alpha+\beta+\gamma}{2}+1)},$$

where V is the region in the first octant bounded by the coordinate planes and the sphere $x^2 + y^2 + z^2 = 1$. [Let $x^2 = u$, $y^2 = v$, and $z^2 = w$ to transform the region of integration into a tetrahedron. After another substitution later, recognize the beta function integral so you can use Theorem 2.2.4.]

Exercise 2.2.3 (Dirichlet Integrals 2). *Show that*

$$\iiint_V x^{\alpha-1}y^{\beta-1}z^{\gamma-1}dV = \frac{a^\alpha b^\beta c^\gamma}{pqr}\frac{\Gamma(\frac{\alpha}{p})\,\Gamma(\frac{\beta}{q})\,\Gamma(\frac{\gamma}{r})}{\Gamma(1+\frac{\alpha}{p}+\frac{\beta}{q}+\frac{\gamma}{r})},$$

where V is the region in the first octant bounded by the coordinate planes and $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r = 1$.

Exercise 2.2.4. *Prove Wallis' Formulas:*

$$\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{(2n)!}{2^{2n}(n!)^2}\frac{\pi}{2} = \frac{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}{2(n!)},$$
$$\int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2^{2n}(n!)^2}{(2n)!}\frac{1}{2n+1} = \frac{\sqrt{\pi}\,n!}{2\,\Gamma(n+\frac{3}{2})}.$$

[Use Exercise 2.1.3, Theorem 2.2.2, and Theorem 2.2.4.]

An interesting fact about Wallis' formulas is that $\frac{(2n)!}{2^{2n}(n!)^2}$ is the probability of getting exactly n heads when $2n$ coins are tossed.

Exercise 2.2.5. *An excellent aproximation to the probability of getting exactly n heads when $2n$ coins are tossed is given by $\frac{1}{\sqrt{n\,\pi}}$. Use Mathematica or Maple to convince yourself that this is true. (The proof will come later.)*

Theorem 2.2.5 (Wallis' Product). $\dfrac{\pi}{2} = \dfrac{2}{1}\cdot\dfrac{2}{3}\cdot\dfrac{4}{3}\cdot\dfrac{4}{5}\cdot\dfrac{6}{5}\cdot\dfrac{6}{7}\cdots\dfrac{2k}{2k-1}\cdot\dfrac{2k}{2k+1}\cdots.$

Proof. Let P_n be the partial product of the first n factors on the right side. We must show that $\lim_{n\to\infty} P_n = \dfrac{\pi}{2}$. From Exercise 2.2.4,

$$\frac{\int_0^{\pi/2}\sin^{2n} x\,dx}{\int_0^{\pi/2}\sin^{2n+1} x\,dx} = \frac{\Gamma(n+\frac{1}{2})}{n!}\cdot\frac{\Gamma(n+\frac{3}{2})}{n!}.$$

By Theorem 2.1.1 and some algebra,

$$\frac{\Gamma(n+\frac{1}{2})}{n!} = \frac{2n-1}{2n}\cdot\frac{2n-3}{2(n-1)}\cdots\frac{3}{2\cdot 2}\cdot\frac{1}{2\cdot 1}\sqrt{\pi},$$
$$\frac{\Gamma(n+\frac{3}{2})}{n!} = \frac{2n+1}{2n}\cdot\frac{2n-1}{2(n-1)}\cdots\frac{3}{2\cdot 1}\cdot\frac{1}{2}\sqrt{\pi}.$$

So the quotient above is

$$\frac{\int_0^{\pi/2}\sin^{2n} x\,dx}{\int_0^{\pi/2}\sin^{2n+1} x\,dx} = \frac{1}{P_{2n}}\cdot\frac{\pi}{2}.$$

Again using Exercise 2.2.4, we have

$$\frac{\int_0^{\pi/2}\sin^{2n+1} x\,dx}{\int_0^{\pi/2}\sin^{2n-1} x\,dx} = \frac{2n}{2n+1},$$

or

$$\int_0^{\pi/2} \sin^{2n-1} x\,dx = \frac{2n+1}{2n} \int_0^{\pi/2} \sin^{2n+1} x\,dx.$$

Since $\sin x$ is increasing and $0 \le \sin x \le 1$ on $[0, \pi/2]$,

$$0 < \int_0^{\pi/2} \sin^{2n+1} x\,dx < \int_0^{\pi/2} \sin^{2n} x\,dx < \int_0^{\pi/2} \sin^{2n-1} x\,dx.$$

Divide by $\int_0^{\pi/2} \sin^{2n+1} x\,dx$ to get

$$1 < \frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx} < \frac{2n+1}{2n}.$$

Clearly, as $n \to \infty$, the middle term $\to 1$, giving us

$$\lim_{n\to\infty} \frac{1}{P_{2n}} \cdot \frac{\pi}{2} = 1 \text{ and so } \lim_{n\to\infty} P_{2n} = \frac{\pi}{2}.$$

Since $\lim_{n\to\infty} P_{2n+1} = \lim_{n\to\infty} \frac{2n+2}{2n+1} P_{2n} = \frac{\pi}{2}$, the proof is complete. ♠

Corollary 2.2.5. $\lim_{n\to\infty} \frac{(2n)!\sqrt{n\pi}}{2^{2n}(n!)^2} = 1.$

Exercise 2.2.6 (Progress as Promised). *Prove Corollary 2.2.5.*

Exercise 2.2.7. *Prove that the approximation in Exercise 2.2.5 is correct by showing that*

$$\frac{(2n)!}{2^{2n}(n!)^2} = \sqrt{1 - \frac{1-\theta_n}{2n+1}} \cdot \frac{1}{\sqrt{n\pi}}$$

for some θ_n satisfying $0 < \theta_n < 1$. [The θ_n comes from using the Mean Value Theorem on one of the inequalities in the proof of Theorem 2.2.5.]

2.3. The Reflection Formula

First, a Fourier series warm-up.

Exercise 2.3.1. *Expand $f(x) = |x|$ for $-\pi \le x \le \pi$, and $f(x + 2\pi) = f(x)$ in Fourier series. Use this result to show that*

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \quad \text{and} \quad \frac{\pi^2}{24} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2}.$$

Now that you have Fourier series back at the top level in your mind, the next exercise will be needed soon.

Exercise 2.3.2. *Expand $f(x) = \cos(zx)$ for $-\pi \le x \le \pi$, and $f(x + 2\pi) = f(x)$ in Fourier series (treat z as a constant) to get*

$$\cos(zx) = \frac{2z}{\pi} \sin(\pi z) \left[\frac{1}{2z^2} + \sum_{k=1}^{\infty} \frac{(-1)^k \cos(kx)}{z^2 - k^2} \right].$$

The following theorem is stated in terms of complex z, but no arguments in the proof require complex analysis, so feel free to think of the z as a real number.

Theorem 2.3.1 (The Reflection Formula). $\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z)$ *for* $0 < \Re(z) < 1$.

Proof. Let $x = 0$ in the series of Exercise 2.3.2 to get

$$\pi \csc(\pi z) = 2z\left[\frac{1}{2z^2} - \frac{1}{z^2-1^2} + \frac{1}{z^2-2^2} - \cdots\right].$$

For $0 < \Re(z) < 1$, Theorems 2.2.3 and 2.2.4 give

$$\begin{aligned}
\Gamma(z)\Gamma(1-z) = B(z, 1-z) &= \int_0^\infty \frac{x^{z-1}}{1+x}dx \\
&= \int_0^1 \frac{x^{z-1}}{1+x}dx + \int_1^\infty \frac{x^{z-1}}{1+x}dx \\
&= \int_0^1 \frac{x^{z-1}}{1+x}dx + \int_1^0 \frac{-t^{-z}}{1+t}dt \\
&= \int_0^1 \frac{x^{z-1}}{1+x}dx + \int_0^1 \frac{x^{-z}}{1+x}dx \\
&= \int_0^1 x^{z-1}dx + \int_0^1 \frac{x^{-z}-x^z}{1+x}dx \\
&= \frac{1}{z} + \int_0^1 (x^{-z}-x^z)(1-x+x^2-x^3+\cdots)\,dx \\
&= \frac{1}{z} + \int_0^1 \sum_{k=0}^\infty (-1)^k(x^{-z+k}-x^{z+k})\,dx \\
&= \frac{1}{z} - \left(\frac{1}{1+z} - \frac{1}{1-z}\right) + \left(\frac{1}{2+z} - \frac{1}{2-z}\right) - \cdots \\
&= 2z\left[\frac{1}{2z^2} - \frac{1}{z^2-1^2} + \frac{1}{z^2-2^2} - \cdots\right].
\end{aligned}$$

The proof is complete provided we can justify the term-by-term integration. Denote by $S_n(x)$ and $R_n(x)$ the n^{th} partial sum and remainder of the series $\sum_{k=0}^\infty (-1)^k x^k(x^{-z}-x^z)$. We need to show that $\int_0^1 R_n(x)\,dx \to 0$ as $n \to \infty$.

$$\int_0^1 |R_n(x)|\,dx = \int_0^1 \frac{x^{n+1}(x^{-z}-x^z)}{1+x}dx = \int_0^1 x^n\left[\frac{x^{1-z}-x^{1+z}}{1+x}\right]dx.$$

Since $0 < z < 1$, the function $\frac{x^{1-z}-x^{1+z}}{1+x}$ is continuous in x on $[0,1]$, and so there is a number M, such that $\dfrac{x^{1-z}-x^{1+z}}{1+x} \le M$ for all $z \in (0,1)$. Thus,

$$\int_0^1 |R_n(x)|\,dx \le \int_0^1 Mx^n\,dx = \frac{M}{n+1}.$$

This completes the proof. ♠

Example 2.3.1 (Another Route to Wallis' Product). *Let* $x = \pi$ *in the series of Exercise 2.3.2 to get*

$$\cos(\pi z) = \frac{2z\sin(\pi z)}{\pi}\left[\frac{1}{2z^2} + \frac{1}{z^2-1^2} + \frac{1}{z^2-2^2} + \frac{1}{z^2-3^2} + \cdots\right],$$

$$\pi \cot(\pi z) - \frac{1}{z} = \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}.$$

Integrate both sides with respect to z from 0 to x, $-1 < x < 1$. (This time term-by-term integration is valid because the series converges uniformly for $|z| < 1$.)

$$\begin{aligned}
\int_0^x \left(\frac{\pi \cos(\pi z)}{\sin(\pi z)} - \frac{1}{z} \right) dz &= \sum_{k=1}^{\infty} \log|z^2 - k^2| \Big]_0^x \\
\log(\sin(\pi x)) - \log x - \lim_{z\to 0} [\log(\sin(\pi z)) - \log z] &= \sum_{k=1}^{\infty} \log\left(\frac{k^2 - x^2}{k^2} \right) \\
\log\left(\frac{\sin(\pi x)}{\pi x} \right) &= \sum_{k=1}^{\infty} \log\left(1 - \frac{x^2}{k^2} \right).
\end{aligned}$$

This is equivalent to

$$\sin(\pi x) = \pi x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2} \right) \quad \textit{for} \ -1 < x < 1.$$

Let $x = \frac{1}{2}$ and factor the term in the product to get

$$\begin{aligned}
1 &= \frac{\pi}{2} \left[\left(\frac{1}{2} \cdot \frac{3}{2} \right) \left(\frac{3}{4} \cdot \frac{5}{4} \right) \left(\frac{5}{6} \cdot \frac{7}{6} \right) \cdots \right] \quad \textit{or} \\
\frac{\pi}{2} &= \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2k}{2k-1} \cdot \frac{2k}{2k+1} \cdots
\end{aligned}$$

In most of the following exercises use Mathematica or Maple to do graphs and numerical work. Proofs are, of course, still your responsibility. If you use Mathematica or Maple to do an integral whose value is given in terms of a special function, think of the result as a theorem to prove.

Exercise 2.3.3. *Evaluate $\int_0^\infty e^{-st}\sqrt{t}\,dt$, which gives the Laplace transform of $\sqrt{t}$.*

Exercise 2.3.4. *Evaluate $\int_0^1 \left(\log\left(\frac{1}{t}\right) \right)^{x-1} dt$ and $\int_0^1 (\log t)^{x-1}\,dt$. When is the second one real-valued?*

Exercise 2.3.5. *Plot the graph of $y = 1/\Gamma(x)$ for $-4 \le x \le 10$. Using the computer, find the first 8 terms in the Taylor series expansion of $1/\Gamma(x)$ around $x = 0$. Do the first 8 terms give a good approximation of the value of $\Gamma(5)$? How about $\Gamma(2)$? Compare with the values from Stirling's formula, and revise, if necessary, your opinion of old Stirling.*

Exercise 2.3.6. *Show that $B(x, x) = 2^{1-2x} B(x, \frac{1}{2})$ for $0 < x < \infty$. Plot the graph of $y = B(x, x)$ and $y = 2^{1-2x} B(x, \frac{1}{2})$ on $(0, 10]$.*

Exercise 2.3.7. *Show that $\sqrt{\pi}\,\Gamma(2x) = 2^{2x-1}\Gamma(x)\,\Gamma(x + \frac{1}{2})$, for $0 < x < \infty$. [Exercise 2.3.6 should help.]*

Exercise 2.3.8. *Evaluate $f(t) = \int_0^{\pi/2} (\sin(2x))^{2t-1} dx$, and plot the graph of f on $(0, 10]$.*

Exercise 2.3.9. *Plot the graph of $x^{2/3} + y^{2/3} = 1$, and find the area inside the curve. [Parameterize the curve in terms of trig functions.]*

2.4. Stirling and Weierstrass

It is a good idea to have a feeling for the order of magnitude of $n!$ and $\Gamma(x)$, especially in comparison with other things that "get real big real fast". The following theorem, due to Stirling, addresses this topic.

Theorem 2.4.1 (Stirling). $\lim_{n\to\infty} \frac{\left(\frac{n}{e}\right)^n \sqrt{2\pi\, n}}{n!} = 1.$

Proof. Let $a_n = \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{n}}$. We will show that $a_n \to \sqrt{2\pi}$ as $n \to \infty$. In Corollary 2.2.5, we can write

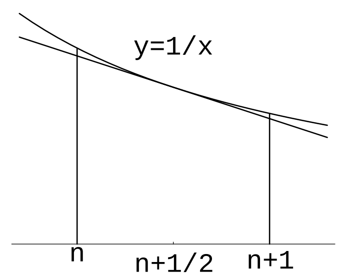
$$\frac{(n!)^2 2^{2n}}{(2n)!\sqrt{n}} = \frac{a_n^2}{\sqrt{2}a_{2n}} = \frac{(n!)^2 \left(\frac{2n}{e}\right)^{2n} \sqrt{2n}}{\sqrt{2}\left(\frac{n}{e}\right)^{2n} n(2n)!}.$$

Assuming $\lim_{n\to\infty} a_n = r \neq 0$, Corollary 2.2.5 gives

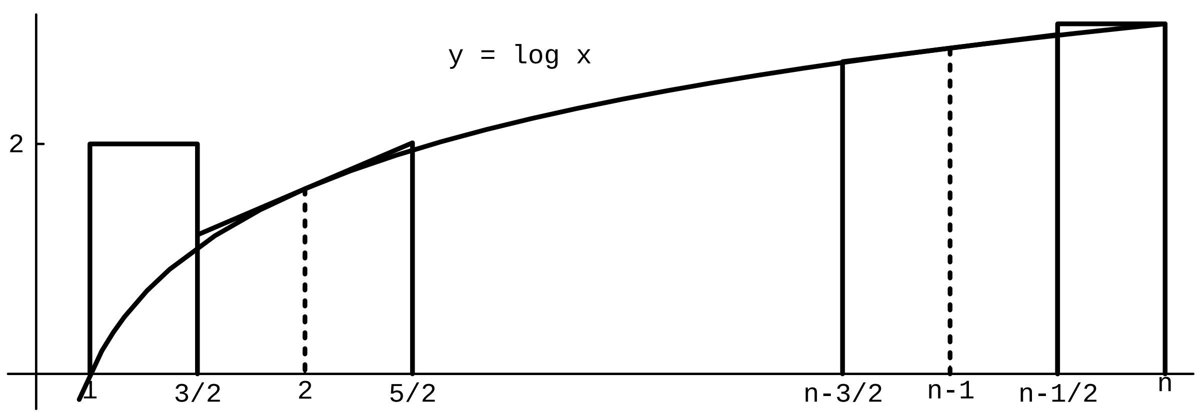
$$\sqrt{\pi} = \frac{r^2}{r\sqrt{2}} \text{ or } r = \sqrt{2\pi}.$$

The proof will be complete when we (you!) show that the sequence $\{a_n\}$ has a nonzero limit. This is done in an exercise. ♠

Exercise 2.4.1. *Show* $\log\left(1 + \frac{1}{n}\right) > \frac{2}{2n+1}$ *for* $n = 1, 2, 3, , \ldots$. *[Hint: See figure.]*



Exercise 2.4.2. *For the sequence $\{a_n\}$ in Theorem 2.4.1 prove that $\{a_n\}$ is a bounded monotonic sequence, and thus has a limit. Further, show that this limit is ≥ 1. [Hint: See figure below.]*



We now prepare for Weierstrass' infinite product representation of the gamma function, which involves Euler's constant, γ.

Lemma 2.4.1. *For* $0 \le x \le 1$, $\lim_{n\to\infty} \dfrac{\Gamma(x+n)}{\Gamma(n)\ n^x} = 1$.

Proof. Since $x \ge 0$ and $x-1 \le 0$, if $0 \le t \le n$, we get $t^x \le n^x$ and $n^{x-1} \le t^{x-1}$ so that

$$n^{x-1} \int_0^n e^{-t} t^n dt \le \int_0^n e^{-t} t^{n+x-1} dt \le n^x \int_0^n e^{-t} t^{n-1} dt. \tag{2.4.1}$$

Similarly, if $n \le t \le \infty$, we have $n^x \le t^x$ and $t^{x-1} \le n^{x-1}$ so that

$$n^x \int_n^\infty e^{-t} t^{n-1} dt \le \int_n^\infty e^{-t} t^{n+x-1} dt \le n^{x-1} \int_n^\infty e^{-t} t^n dt. \tag{2.4.2}$$

In (2.4.2) integrate the outside integrals by parts to get

$$-e^{-n} n^{n+x-1} + n^{x-1} \int_n^\infty e^{-t} t^n dt \le \int_n^\infty e^{-t} t^{n+x-1} dt \le e^{-n} n^{n+x-1} + n^x \int_n^\infty e^{-t} t^{n-1} dt. \tag{2.4.3}$$

Add (2.4.1) and (2.4.3) and note the appearance of gamma functions to get

$$-e^{-n} n^{n+x-1} + n^{x-1}\Gamma(n+1) \le \Gamma(x+n) \le e^{-n} n^{n+x-1} + n^x \Gamma(n).$$

Divide by $n^x\Gamma(n)$ and simplify to get

$$-\frac{e^{-n}n^n}{n!} + 1 \le \frac{\Gamma(x+n)}{\Gamma(n)\ n^x} \le \frac{e^{-n}n^n}{n!} + 1.$$

By Theorem 2.4.1 (Stirling) $\lim_{n\to\infty} \dfrac{e^{-n}n^n}{n!} = 0$, which completes the proof. ♠

Lemma 2.4.2. *For* $0 \le x < \infty$, $\lim_{n\to\infty} \dfrac{\Gamma(x+n)}{\Gamma(n)\ n^x} = 1$.

Sketch of Proof. Use induction and the fact that

$$\frac{\Gamma(x+n)}{\Gamma(n)\ n^x} = \frac{x-1+n}{n} \cdot \frac{\Gamma(x-1+n)}{\Gamma(n)\ n^{x-1}}. \ ♠$$

The following result of Weierstrass can be, and sometimes is, used to *define* the gamma function instead of the integral in Definition 2.1.1. See, for example, *A Course of Modern Analysis* by Whittaker and Watson.

Theorem 2.4.2 (Weierstrass). *If* $x > 0$ *and* γ *denotes Euler's constant, then*

$$\frac{1}{\Gamma(x)} = x\, e^{\gamma x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right) e^{-x/k}.$$

Proof. Let

$$P_n = x \prod_{k=1}^{n-1} \left(1 + \frac{x}{k}\right) e^{-x/k} = \left(x \prod_{k=1}^{n-1} (x+k)\right) \left(\prod_{k=1}^{n-1} \frac{1}{k}\right) \left(\prod_{k=1}^{n-1} e^{-x/k}\right) = \left(\prod_{k=0}^{n-1} (x+k)\right) \frac{1}{\Gamma(n)} e^{-xH_{n-1}},$$

where $H_{n-1} = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}$. By Theorem 2.1.1,

$$\begin{aligned} \frac{1}{\Gamma(x)} &= \frac{\prod_{k=0}^{n-1}(x+k)}{\Gamma(x+n)} = \frac{P_n \Gamma(n) e^{xH_{n-1}}}{\Gamma(x+n)} \cdot \frac{n^x}{n^x} \\ &= \frac{\Gamma(n)\ n^x}{\Gamma(x+n)} P_n e^{xH_{n-1}} e^{-x\log n} = \frac{\Gamma(n)\ n^x}{\Gamma(x+n)} P_n e^{(H_{n-1} - \log n)x}. \end{aligned}$$

By the definition of γ and Lemmas 2.4.1 and 2.4.2, we get $\lim_{n\to\infty} P_n = \dfrac{e^{-\gamma x}}{\Gamma(x)}$. ♠

Weierstrass' theorem connects the gamma function and Euler's constant. This connection can be further exploited.

Theorem 2.4.3. $\Gamma'(1) = -\gamma$.

Proof. The logarithmic derivative of the gamma function, i.e., the derivative of $\log(\Gamma(x))$, is called the *digamma function* and is denoted by $\psi(x)$. Taking logs and differentiating in Theorem 2.4.2 gives

$$-\psi(x) = \gamma + \frac{1}{x} - \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+x}\right),$$

and, for $x = 1$,

$$-\psi(1) = \gamma + 1 - \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \gamma.$$

Since $\psi(1) = \frac{\Gamma'(1)}{\Gamma(1)}$ and $\Gamma(1) = 1$, the proof is complete. ♠

Exercise 2.4.3. *Show that* $\psi(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k}$ *for* $n \geq 2$ *and an integer. Find* $\Gamma'(2)$, $\Gamma'(3)$, *and* $\Gamma'(17)$.

Exercise 2.4.4. *Show that* $\psi(\frac{1}{2}) = -\gamma - 2\log 2$ *and* $\psi(n+\frac{1}{2}) = -\gamma - 2\log 2 + 2\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right)$ *for* $n \geq 1$ *and an integer.*

Exercise 2.4.5 (Difference Equation). *Show that* $\psi(x+1) = \psi(x) + \frac{1}{x}$.

2.5. Evaluation of a Class of Infinite Products

Suppose u_n is a rational function of n written as

$$u_n = \frac{A(n-a_1)(n-a_2)\cdots(n-a_k)}{(n-b_1)(n-b_2)\cdots(n-b_j)}.$$

In order for the product $\prod_{n=1}^{\infty} u_n$ to converge absolutely, we need $A = 1$ and $j = k$, because otherwise $u_n \not\to 1$ as $n \to \infty$. Thus, we are led to the product

$$P = \prod_{n=1}^{\infty} u_n = \prod_{n=1}^{\infty} \frac{(n-a_1)(n-a_2)\cdots(n-a_k)}{(n-b_1)(n-b_2)\cdots(n-b_k)}$$

where the general term u_n can be written as

$$\begin{aligned} u_n &= \left(1 - \frac{a_1}{n}\right)\cdots\left(1 - \frac{a_k}{n}\right)\left(1 - \frac{b_1}{n}\right)^{-1}\cdots\left(1 - \frac{b_k}{n}\right)^{-1} \\ &= 1 - \frac{a_1 + a_2 + \cdots + a_k - b_1 - b_2 - \cdots - b_k}{n} + O(n^{-2}) \end{aligned}$$

where the Binomial Theorem was used to expand the negative powers. Absolute convergence forces the $\frac{1}{n}$ term to be 0, or $a_1 + \cdots + a_k - b_1 - \cdots - b_k = 0$. Thus, $\exp\left(\frac{a_1+a_2+\cdots+a_k-b_1-b_2-\cdots-b_k}{n}\right) = 1$, and can multiply u_n without changing P.

$$P = \prod_{n=1}^{\infty} \frac{\left(1 - \frac{a_1}{n}\right)e^{\frac{a_1}{n}}\left(1 - \frac{a_2}{n}\right)e^{\frac{a_2}{n}}\cdots\left(1 - \frac{a_k}{n}\right)e^{\frac{a_k}{n}}}{\left(1 - \frac{b_1}{n}\right)e^{\frac{b_1}{n}}\left(1 - \frac{b_2}{n}\right)e^{\frac{b_2}{n}}\cdots\left(1 - \frac{b_k}{n}\right)e^{\frac{b_k}{n}}}$$

Now use Theorem 2.4.2 to express P as

$$P = \prod_{n=1}^{\infty} \frac{(n-a_1)(n-a_2)\cdots(n-a_k)}{(n-b_1)(n-b_2)\cdots(n-b_k)} = \prod_{i=1}^{k} \frac{\Gamma(1-b_i)}{\Gamma(1-a_i)}. \tag{2.5.1}$$

Exercise 2.5.1. *Fill in all the details in the derivation of equation (2.5.1).*

Exercise 2.5.2. *Evaluate, if possible,* $\displaystyle\prod_{n=1}^{\infty} \frac{(n+2)(n+5)(n+7)}{(n+4)^2(n+6)}$.

Example 2.5.1. *Evaluate* $x\left(1-\frac{x}{1^n}\right)\left(1-\frac{x}{2^n}\right)\left(1-\frac{x}{3^n}\right)\cdots$*, where n is a positive integer.*

$$P = x\prod_{k=1}^{\infty}\left(1-\frac{x}{k^n}\right) = x\prod_{k=1}^{\infty}\left(\frac{k^n-(x^{1/n})^n}{k^n}\right).$$

Let $\alpha = e^{\frac{2\pi i}{n}}$ so that $\alpha^n = e^{2\pi i} = 1$. Note that the n n^{th} roots of 1 are $\alpha^0, \alpha^1, \ldots, \alpha^{n-1}$. Let $z = x^{1/n}$. Then $k^n - z^n = (k-\alpha^0 z)(k-\alpha^1 z)\cdots(k-\alpha^{n-1}z)$. (Remember that the "unknown" is k, the product index.) We can now write

$$P = z^n \prod_{k=1}^{\infty} \frac{(k-\alpha^0 z)(k-\alpha^1 z)\cdots(k-\alpha^{n-1}z)}{(k-0)(k-0)\cdots(k-0)}.$$

Clearly, $b_1+b_2+\cdots+b_n = 0$. The sum $a_1+a_2+\cdots+a_n$ is the same as $z(\alpha^0+\alpha^1+\cdots+\alpha^{n-1})$, and since the α^j's are the roots of a polynomial of degree n with no degree $(n-1)$ term, their sum is 0. Hence the product is absolutely convergent, and equation (2.5.1) may be applied to get

$$P = z^n \prod_{j=0}^{n-1} \frac{\Gamma(1)}{\Gamma(1-\alpha^j z)} = z^n \frac{1}{-\alpha^0 z\Gamma(-\alpha^0 z)(-\alpha^1 z)\Gamma(-\alpha^1 z)\cdots(-\alpha^{n-1}z)\Gamma(-\alpha^{n-1}z)}$$

$$= \frac{1}{(-1)^n\alpha^{1+2+\cdots+(n-1)}\Gamma(-x^{1/n})\Gamma(-\alpha^1 x^{1/n})\cdots\Gamma(-\alpha^{n-1}x^{1/n}}$$

Since $\alpha^{1+2+\cdots+(n-1)} = (-1)^{n-1}$, we get

$$P = \frac{1}{-\Gamma(-x^{1/n})\Gamma(-\alpha^1 x^{1/n})\cdots\Gamma(-\alpha^{n-1}x^{1/n})}.$$

Exercise 2.5.3. *Evaluate* $(1-z)\left(1+\frac{z}{2}\right)\left(1-\frac{z}{3}\right)\left(1+\frac{z}{4}\right)\cdots$.

Exercise 2.5.4. *Evaluate* $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$, $\prod_{n=k+1}^{\infty}\left(1-\frac{k^2}{n^2}\right)$*, and* $\prod_{n=k+1}^{\infty}\left(1-\frac{k^m}{n^m}\right)$*. In the last product, m is a positive integer.*

Exercise 2.5.5. *Define the Cosine Integral Function $Ci(x)$, by*

$$Ci(x) = -\int_x^{\infty} \frac{\cos t}{t}dt.$$

Show that $\lim_{x\to\infty} Ci(x) = 0$*. (See A&S, Chapter 5.)*

Exercise 2.5.6. *(American Mathematical Monthly, 1998, pp. 278-279) Show that*

$$\gamma = \lim_{u\to\infty}\int_{1/u}^{u}\left(\frac{1}{2}-\cos x\right)\frac{dx}{x} = \lim_{u\to\infty}\int_{1/u}^{u}\left(\frac{1}{1+x}-\frac{\cos x}{x}\right)dx.$$