Special Functions

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Chapter 4. Hypergeometric Functions

4.1. Solutions of Linear DEs at Regular Singular Points

Consider the differential equation

$$y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0. \tag{4.1.1}$$

If either p or q has a singularity at $x = x_0$, then x_0 is a *singular point* of (4.1.1). The singular point x_0 is *regular* if both the limits

$$\lim_{x\to x_0}(x - x_0)p(x) \quad \text{and} \quad \lim_{x\to x_0}(x - x_0)^2 q(x)$$

exist. Call these limits, when they exist, p_0 and q_0. The *exponents at the regular singular point* x_0 of (4.1.1) are the roots of the *indicial equation*

$$r(r-1) + p_0 r + q_0 = 0.$$

If x_0 is a regular singular point of (4.1.1), then one solution is representable as a *Frobenius series* and has the form

$$y_1(x) = \sum_{k=0}^{\infty} a_k(r)(x - x_0)^{k+r} \tag{4.1.2}$$

where r is an exponent at x_0, i.e., a root of the indicial equation. The coefficients a_k can be found up to a constant multiple by substituting the series into (4.1.1) and equating coefficients. Unfortunately, we are only guaranteed one Frobenius series solution of (4.1.1), which is a second order linear homogeneous DE, and so has two linearly independent solutions. The second solution in the neighborhood of a regular singular point will take one of three forms.

Case 1. *If the exponents do not differ by an integer, then the second solution of (4.1.1) is found by using the other exponent in the series (4.1.2).*

Case 2. *If the exponents are equal, the second solution has the form*

$$y_2(x) = y_1(x)\,\log(x - x_0) + \sum_{k=1}^{\infty} b_k(r)\,(x - x_0)^{k+r},$$

where r is the exponent and y_1 is the solution given by (4.1.2).

Case 3. *If the exponents r_1 and r_2 differ by a positive integer, $r_1 - r_2 = N$, then one solution is given by (4.1.2) using $r = r_1$, and the second solution has the form*

$$y_2(x) = C\,y_1(x)\,\log(x - x_0) + \sum_{k=0}^{\infty} c_k(r_2)\,(x - x_0)^{k+r_2}.$$

The constant C may or may not be zero.

Example 4.1.1. *Legendre's differential equation is*

$$(1 - x^2)y''(x) - 2x\,y'(x) + n(n+1)\,y(x) = 0.$$

The most interesting case is when n is a nonnegative integer. At the regular singular point $x = 1$, the indicial equation is $r^2 = 0$, making the exponents at $x = 1$ equal to $0, 0$. For simplicity using Frobenius series, translate $x = 1$ to the origin by $x = u + 1$. The equivalent DE is

$$u(u+2)\,y''(u) + 2(u+1)\,y'(u) - n(n+1)\,y(u) = 0.$$

The regular singular point $u = 0$ corresponds to $x = 1$ and has the same exponents, both 0. The Frobenius series is $\sum_{k=0}^{\infty} a_k u^k$, and substitution of the series into the DE yields

$$\sum_{k=0}^{\infty} \left[(k+n+1)(k-n)a_k + 2(k+1)^2 a_{k+1}\right] u^k = 0.$$

Equating coefficients leads to the recurrence relation

$$a_{k+1} = \frac{-(k+n+1)(k-n)}{2(k+1)^2} a_k,$$

which gives

$$a_k = \frac{(-1)^k (n+1)_k (-n)_k}{2^k (k!)^2} a_0.$$

The the $(\cdot)_k$ notation represents the factorial function and is defined by $(z)_k = z(z+1)\cdots(z+k-1) = \Gamma(z+k)/\Gamma(z)$. The Frobenius series solution to Legendre's DE is, for $a_0 = 1$,

$$y_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-n)_k (n+1)_k}{(1)_k k!} \left(\frac{1-x}{2}\right)^k.$$

Note that the series terminates if n is a nonnegative integer; the resulting polynomial is denoted $P_n(x)$, and is the Legendre polynomial of degree n. Also note that $P_n(1) = 1$ for all nonnegative integers n.

Exercise 4.1.1. *Fill in all the details and verify all the claims in Example 4.1.1. Get comfortable dealing with the factorial function.*

Example 4.1.2. *Bessel's differential equation is*

$$x^2\, y''(x) + x\, y'(x) + (x^2 - \nu^2)\, y(x) = 0.$$

The Frobenius series solution turns out to be

$$y_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k} k! (1+\nu)_k} a_0.$$

If we let $a_0 = \dfrac{1}{2^\nu \Gamma(\nu+1)}$, we get the "standard" solution to Bessel's DE, the Bessel funtion of the first kind of order ν, denoted by $J_\nu(x)$:

$$J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{k!(1+\nu)_k} \left(\frac{-x^2}{4}\right)^k.$$

Exercise 4.1.2. *For Bessel's DE, show that $x = 0$ is a regular singular point with exponents $\pm\nu$, and fill in the details in the derivation of the formula for $J_\nu(x)$.*

4.2. Equations of Fuchsian Type

Consider the differential equation

$$y''(z) + p(z)\, y'(z) + q(z)\, y(z) = 0. \tag{4.2.1}$$

We call (4.2.1) an *equation of Fuchsian type* if every singular point is a regular singular point.

Lemma 4.2.1. *If (4.2.1) is of Fuchsian type, then the number of singular points of (4.2.1) is finite.*

Proof. At each singular point, either p or q has a pole. Suppose there are infinitely many singular points. Then either p or q has infinitely many poles. These poles have a limit point (possibly ∞) which is an essential singularity of p or q. But such an essential singularity corresponds to an irregular singular point of (4.2.1), contradicting the assumption that (4.2.1) is of Fuchsian type. ♠

Suppose (4.2.1) is of Fuchsian type and has exactly $m+1$ distinct singular points, where $m \geq 2$. Denote the singularities by $z = z_k, \quad k = 1, \ldots, m \quad$ and $\quad z = \infty$. Then p can have no singularities in the finite plane except poles of order one at the $z_k s$. So,

$$p(z) = \frac{p_1(z)}{(z-z_1)(z-z_2)\cdots(z-z_m)},$$

where p_1 is a polynomial. Also, q can have no singularities except poles of order $\leq$ two at the $z_k s$:

$$q(z) = \frac{q_1(z)}{(z-z_1)^2(z-z_2)^2\cdots(z-z_m)^2}.$$

The maximum degree of the polynomials p_1 and q_1 can be determined using the regular singular point at ∞. Let $z = 1/t$, giving

$$\frac{d^2y}{dt^2} + \frac{1}{t}\left[2 - \frac{p_1(\frac{1}{t})}{t(\frac{1}{t}-z_1)\cdots(\frac{1}{t}-z_m)}\right]\frac{dy}{dt} + \frac{1}{t^2}\left[\frac{q_1(\frac{1}{t})}{t^2(\frac{1}{t}-z_1)^2\cdots(\frac{1}{t}-z_m)^2}\right]y = 0. \tag{4.2.2}$$

In order for $z = \infty$, or $t = 0$, to be a regular singular point, the functions in the brackets in (4.2.2) must be analytic at $t = 0$. This means $degree(p_1) \leq m-1$ and $degree(q_1) \leq 2m-2$. Thus we have the following theorem.

Theorem 4.2.1. *If equation (4.2.1) is of Fuchsian type and has exactly $m+1$ distinct singular points, $z = z_k, \quad k = 1, \ldots, m \quad$ and $\quad z = \infty$, then (4.2.1) can be written*

$$y''(z) + \frac{T_{(m-1)}(z)}{\psi(z)}\,y'(z) + \frac{T_{(2m-2)}(z)}{\psi^2(z)}\,y(z) = 0,$$

where $\psi(z) = \prod_{k=1}^{m}(z-z_k)$ and $T_{(j)}(z)$ is a polynomial of degree at most j in z.

Corollary 4.2.1. *There exist constants A_k, $k = 1, 2, \ldots, m$ such that $p(z) = \sum_{k=1}^{m} \frac{A_k}{z-z_k}$.*

Corollary 4.2.2. *There exist constants B_k and C_k, $k = 1, 2, \ldots, m$, such that*

$$q(z) = \sum_{k=1}^{m}\left(\frac{B_k}{(z-z_k)^2} + \frac{C_k}{z-z_k}\right), \text{ and } \sum_{k=1}^{m} C_k = 0.$$

Exercise 4.2.1. *Prove both the above corollaries.*

Equation (4.2.1) thus can be written in the form

$$y''(z) + \sum_{k=1}^{m}\frac{A_k}{z-z_k}\,y'(z) + \sum_{k=1}^{m}\left(\frac{B_k}{(z-z_k)^2} + \frac{C_k}{z-z_k}\right)y(z) = 0, \tag{4.2.3}$$

where $\sum_{k=1}^{m} C_k = 0$.

Denote the exponents at the singular point z_k by $\alpha_{1,k}$ and $\alpha_{2,k}$, and the exponents at ∞ by $\alpha_{1,\infty}$ and $\alpha_{2,\infty}$. Since the indicial equation at z_k is $r^2 + (A_k - 1)r + B_k = 0$, we get

$$\alpha_{1,k} + \alpha_{2,k} = 1 - A_k \quad \text{and} \quad \alpha_{1,k}\alpha_{2,k} = B_k.$$

For the singularity at ∞, we get

$$\alpha_{1,\infty} + \alpha_{2,\infty} = -1 + \sum_{k=1}^{m} A_k \quad \text{and} \quad \alpha_{1,\infty}\alpha_{2,\infty} = \sum_{k=1}^{m} (B_k + C_k z_k).$$

Thus, the sum of all the exponents for all the singular points is

$$\alpha_{1,\infty} + \alpha_{2,\infty} + \sum_{k=1}^{m} (\alpha_{1,k} + \alpha_{2,k}) = m - 1.$$

This number depends only on the number of singularities (and the order of the equation), and is the *Fuchsian invariant* for the second order DE of Fuchsian type. For the Fuchsian DE of order n, the Fuchsian invariant is $(m-1)n(n-1)/2$.

Example 4.2.1. *A second order Fuchsian DE with $m = 2$ contains five arbitrary parameters, A_1, A_2, B_1, B_2, and $C_1 = -C_2$. Also, there are six exponents, with sum one (Fuchsian invariant), such that*

$$\begin{aligned}
&A_1 = 1 - \alpha_{1,1} - \alpha_{2,1} \\
&A_2 = 1 - \alpha_{1,2} - \alpha_{2,2} \\
&B_1 = \alpha_{1,1}\alpha_{2,1} \\
&B_2 = \alpha_{1,2}\alpha_{2,2} \\
&B_1 + B_2 + C_1 z_1 + C_2 z_2 = \alpha_{1,\infty}\alpha_{2,\infty} \\
&C_1 + C_2 = 0
\end{aligned}$$

These relationships allow us to write (4.2.3) in terms of the exponents.

$$\begin{aligned}
y''(z) + &\left[\frac{1 - \alpha_{1,1} - \alpha_{2,1}}{z - z_1} + \frac{1 - \alpha_{1,2} - \alpha_{2,2}}{z - z_2} \right] y'(z) + \\
&\left[\frac{\alpha_{1,1}\alpha_{2,1}}{(z - z_1)^2} + \frac{\alpha_{1,2}\alpha_{2,2}}{(z - z_2)^2} + \frac{\alpha_{1,\infty}\alpha_{2,\infty} - \alpha_{1,1},\alpha_{2,1} - \alpha_{1,2}\alpha_{2,2}}{(z - z_1)(z - z_2)} \right] y(z) = 0. \qquad (4.2.4)
\end{aligned}$$

4.3. The Riemann-Papperitz Equation

Now assume (4.2.1) has three regular singular points, all finite, and that ∞ is an ordinary point. Denote the singularities by a, b, and c and denote the corresponding exponents by a' and a'', b' and b'', and c' and c''. Equation (4.2.1) then has the form

$$y''(z) + \frac{p_2(z)}{(z-a)(z-b)(z-c)} y'(z) + \frac{q_2(z)}{(z-a)^2(z-b)^2(z-c)^2} y(z) = 0. \qquad (4.3.1)$$

Exercise 4.3.1. *Use the fact that ∞ is an ordinary point of (4.3.1) to show that: (i) p_2 is a polynomial of degree two with the coefficient of z^2 equal to 2; (ii) q_2 is a polynomial of degree ≤ 2.*

From Exercise 4.3.1, we see that there exist constants A_1, A_2, A_3, B_1, B_2, and B_3 such that

$$\begin{aligned}
\frac{p_2(z)}{(z-a)(z-b)(z-c)} &= \frac{A_1}{z-a} + \frac{A_2}{z-b} + \frac{A_3}{z-c}, \\
\frac{q_2(z)}{(z-a)(z-b)(z-c)} &= \frac{B_1}{z-a} + \frac{B_2}{z-b} + \frac{B_3}{z-c}
\end{aligned},$$

and $A_1 + A_2 + A_3 = 2$. The form of the DE is now

$$y''(z) + \left[\frac{A_1}{z-a} + \frac{A_2}{z-b} + \frac{A_3}{z-c}\right] y'(z) + \left[\frac{B_1}{z-a} + \frac{B_2}{z-b} + \frac{B_3}{z-c}\right] \frac{y(z)}{(z-a)(z-b)(z-c)} = 0. \tag{4.3.2}$$

Exercise 4.3.2. *Using the indicial equations for (4.3.2), show that*

$$\begin{aligned}
a' + a'' &= 1 - A_1 \\
b' + b'' &= 1 - A_2 \\
c' + c'' &= 1 - A_3 \\
a'a'' &= \frac{B_1}{(a-b)(a-c)} \\
b'b'' &= \frac{B_2}{(b-a)(b-c)} \\
c'c'' &= \frac{B_3}{(c-a)(c-b)} \\
a' + a'' + b' + b'' + c' + c'' &= 1
\end{aligned}$$

So, in terms of the exponents, (4.3.2) becomes

$$\begin{aligned}
y''(z) + &\left[\frac{1-a'-a''}{z-a} + \frac{1-b'-b''}{z-b} + \frac{1-c'-c''}{z-c}\right] y'(z) + \\
&\left[\frac{a'a''(a-b)(a-c)}{z-a} + \frac{b'b''(b-a)(b-c)}{z-b} + \frac{c'c''(c-a)(c-b)}{z-c}\right] \frac{y(z)}{(z-a)(z-b)(z-c)} = 0
\end{aligned} \tag{4.3.3}$$

This is the *Riemann-Papperitz* equation. If y is a solution of the Riemann-Papperitz equation, we use the *Riemann P-function* notation

$$y = P \begin{pmatrix} a & b & c & \\ a' & b' & c' & z \\ a'' & b'' & c'' & \end{pmatrix}.$$

The right side is simply a symbol used to explicitly exhibit the singularities and their exponents. If c is replaced by ∞ then y satisfies the DE with $c \to \infty$. It can be shown that the results agree with what we got in the last section.

There are two useful properties of the Riemann P-function we will need later.

Theorem 4.3.1. *If a linear fractional transformation of the form*

$$z = \frac{At+B}{Ct+D} = \frac{(a-b)(a-c)(c_1-b_1)(t-a_1)}{(a-b)(a_1-c_1)(t-b_1)+(a-c)(b_1-a_1)(t-c_1)} + a$$

transforms a, b, and c into a_1, b_1, and c_1 respectively, then

$$P \begin{pmatrix} a & b & c & \\ a' & b' & c' & z \\ a'' & b'' & c'' & \end{pmatrix} = P \begin{pmatrix} a_1 & b_1 & c_1 & \\ a' & b' & c' & t \\ a'' & b'' & c'' & \end{pmatrix}.$$

This can be verified by direct, but tedious, substitution.

Theorem 4.3.2.

$$P\begin{pmatrix} a & b & c & \\ a' & b' & c' & z \\ a'' & b'' & c'' & \end{pmatrix} = \left(\frac{z-a}{z-b}\right)^k P\begin{pmatrix} a & b & c & \\ a'-k & b'+k & c' & z \\ a''-k & b''+k & c'' & \end{pmatrix}.$$

Outline of Proof. If $w(z)$ satisfies (4.3.3), let $w(z) = \left(\frac{z-a}{z-b}\right)^k w_1(z)$. We will show that w_1 satisfies an equation of the form (4.3.3), but with the exponent a' replaced by $a'-k$. Corresponding to the regular singular point $z=a$, there is a Frobenius series solution corresponding to the exponent a':

$$w = \sum_{n=0}^{\infty} a_n\,(z-a)^{n+a'}.$$

Thus,

$$w_1(z) = (z-b)^k \sum_{n=0}^{\infty} a_n\,(z-a)^{n+a'-k}.$$

But $(z-b)^k$ is analytic at $z=a$, and has a series expansion around $z=a$

$$(z-b)^k = (a-b)^k + \sum_{n=1}^{\infty} b_n\,(z-a)^n,$$

so we can write

$$w_1(z) = \sum_{n=0}^{\infty} c_n\,(z-a)^{n+a'-k}$$

where $c_0 \neq 0$. Thus, the a' in the symbol for the Riemann P-function for w becomes $a'-k$ in the symbol for w_1. The other three exponents are similar. ♠

Exercise 4.3.3. *The transformation $y(z) = f(z)\,v(z)$ applied to (4.2.1) yields*

$$v'' + \left(2\frac{f'}{f} + p\right)v' + \left(\frac{f''}{f} + p\frac{f'}{f} + q\right)v = 0.$$

Also, $\frac{f'}{f} = (\log f)'$ and $\frac{f''}{f} = \left(\frac{f'}{f}\right)' + \left(\frac{f'}{f}\right)^2$. Now apply $w(z) = \left(\frac{z-a}{z-b}\right)^k w_1(z)$ to (4.3.3). Show that the indicial equation at $z=a$ is transformed from

$$r^2 - (a'+a'')\,r + a'a'' = 0$$

into

$$r^2 - (a'+a''-2k)\,r + k^2 - (a'+a'')\,k + a'a'' = 0.$$

Based on this work, prove Theorem 4.3.2.

4.4. The Hypergeometric Equation

Theorems 4.3.1 and 4.3.2 can be used to reduce (4.3.3) to a simple canonical form. Let $w(z)$ be a solution of (4.3.3) as before and let $w(z) = \left(\frac{z-a}{z-b}\right)^{a'} w_1(z)$. Then by Theorem 4.3.2, w_1 is a solution of the DE corresponding to

$$P\begin{pmatrix} a & b & c & \\ 0 & b'+a' & c' & z \\ a''-a' & b''+a' & c'' & \end{pmatrix}$$

Another zero exponent can be obtained by letting $w_1(z) = \left(\dfrac{z-b}{z-c}\right)^{b'+a'} w_2(z)$ so that w_2 is represented by

$$P\left(\begin{array}{cccc} a & b & c & \\ 0 & 0 & c'+b'+a' & z \\ a''-a' & b''-b' & c''+b'+a' & \end{array}\right)$$

Note that the sum of the six exponents is still 1. Now let $\alpha = a'+b'+c'$, $\beta = a'+b'+c''$, and $\gamma = 1-a''+a'$. The Riemann P-function representing w_2 is now

$$P\left(\begin{array}{cccc} a & b & c & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{array}\right).$$

Penultimately, use a linear fractional transformation to map a, b, and c to 0, 1, and ∞ respectively:

$$t = \frac{(b-c)(z-a)}{(b-a)(z-c)}.$$

Finally, rename t to be z. We have the Riemann P-function $P\left(\begin{array}{cccc} 0 & 1 & \infty & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{array}\right)$, which corresponds to the *hypergeometric DE*:

$$z(1-z)y''(z) + [\gamma - (\alpha+\beta+1)z]\, y'(z) - \alpha\,\beta\, y(z) = 0 \tag{4.4.1}$$

Exercise 4.4.1. *Fill in the details in the derivation of (4.4.1).*

Since (4.4.1) has a regular singular point at $z = 0$ with one exponent 0, one solution has the form

$$y = \sum_{k=0}^{\infty} a_k\, z^k$$

and the usual series manipulations lead to the recurrence relation

$$a_k = \frac{(k-1+\alpha)(k-1+\beta)}{k(k-1+\gamma)}\, a_{k-1}.$$

If we set $a_0 = 1$, we get the *hypergeometric function* $F(\alpha, \beta; \gamma; z)$ as a solution.

$$F(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k\, k!} z^k,$$

provided $\gamma \neq 0, -1, -2, \ldots$. If we also assume $\gamma \neq 1, 2, 3, \ldots$ the solution around $z = 0$ corresponding to the other exponent, $1-\gamma$, is

$$y_2(z) = \sum_{k=0}^{\infty} \frac{(1-\gamma+\alpha)_k (1-\gamma+\beta)_k}{(2-\gamma)_k\, k!} z^{k+1-\gamma} = z^{1-\gamma} F(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma; z)$$

Many known functions can be expressed in terms of the hypergeometric function. Here are some examples.

Example 4.4.1. *Polynomials. If either α or β is zero or a negative integer the series terminates.*

$$F(\alpha,0;\gamma;z)=1, \qquad F(\alpha,-n;\gamma;z)=\sum_{k=0}^{n}\frac{(\alpha)_k(-n)_k}{(\gamma)_k\,k!}\,z^k.$$

Example 4.4.2. *Logarithms.* $z\,F(1,1;2;-z)=\log(1+z)$ *and* $2z\,F(\frac{1}{2},1;\frac{3}{2};z^2)=\log\frac{1+z}{1-z}$.

Example 4.4.3. *Inverse trigonometric functions.*
$z\,F(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2)=\arcsin z$ *and* $z\,F(\frac{1}{2},1;\frac{3}{2};-z^2)=\arctan z$.

Example 4.4.4. *Rational functions and/or binomial expansions.* $F(\alpha,\beta;\beta;z)=\frac{1}{(1-z)^\alpha}=(1-z)^{-\alpha}$.

Example 4.4.5. *Complete elliptic integrals. In the following, z is the modulus, not the parameter.*
$K(z)=\frac{\pi}{2}\,F(\frac{1}{2},\frac{1}{2};1;z^2),$ *and* $E(z)=\frac{\pi}{2}\,F(-\frac{1}{2},\frac{1}{2};1;z^2).$
(See Math Mag. 68(3), June 1995, p.216 for an article on the rate of convergence of these hypergeometric functions.)

Exercises 4.4.2-6. *Verify the claims in Examples 4.4.1-5.*

Example 4.4.6. *Legendre polynomials. For n a positive integer, $P_n(z)=F(-n,n+1;1;\frac{1-z}{2})$. This can be seen from the form of the series solution (see Example 4.1.1)), or can be derived directly from Legendre's DE, $(1-z^2)y''(z)-2z\,y'(z)+n(n+1)\,y(z)=0$, n a positive integer. The regular singular points are at ±1 and ∞, and the transformation $t=\frac{1-z}{2}$ takes $1\to0$, $-1\to1$, and $\infty\to\infty$. The DE becomes*

$$t(1-t)y''(t)+(1-2t)y'(t)+n(n+1)y(t)=0,$$

which can be seen to be the hypergeometric DE in t with $\alpha=-n$, $\beta=n+1$, and $\gamma=1$.

Exercise 4.4.7. $F(\alpha,\beta;\gamma;z)=F(\beta,\alpha;\gamma;z)$.

Exercise 4.4.8. $\dfrac{d}{dz}\,F(\alpha,\beta;\gamma;z)=\dfrac{\alpha\,\beta}{\gamma}\,F(\alpha+1,\beta+1;\gamma+1;z).$

Hypergeometric functions in which α, β, or γ are replaced by $\alpha\pm1$, $\beta\pm1$, or $\gamma\pm1$ are called *contiguous* to $F(\alpha,\beta;\gamma;z)$. Gauss proved that $F(\alpha,\beta;\gamma;z)$ and any two of its contiguous functions are related by a linear relation with coefficients linear functions of z. The following exercises illustrate two such relations. There are many more.

Exercise 4.4.9. $(\gamma-\alpha-\beta)\,F(\alpha,\beta;\gamma;z)+\alpha(1-z)\,F(\alpha+1,\beta;\gamma;z)-(\gamma-\beta)\,F(\alpha,\beta-1;\gamma;z)=0.$

Exercise 4.4.10. $F(\alpha,\beta+1;\gamma;z)-F(\alpha,\beta;\gamma;z)=\frac{\alpha\,z}{\gamma}F(\alpha+1,\beta+1;\gamma+1;z).$

The infinite product result in Section 2.5 can be used to evaluate $F(\alpha,\beta;\gamma;1)$ in terms of gamma functions. Details can be found in Whittaker and Watson, pp. 281-2. Limits are necessary because $z=1$ is a singular point of the hypergeometric differential equation.

$$F(\alpha,\beta;\gamma;1)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}.$$

4.5. Confluence of Singularities

Many differential equations of interest have an irregular singular point. The harmonic oscillator equation, $y'' + y = 0$, has an irregular singularity at ∞, for example. The results for DEs of Fuchsian type can be used to study such equations under the right circumstances. We may let two singularities come together and become an irregular singular point provided: (1) at least one of the corresponding exponents approaches ∞ and (2) the DE has a limiting form. This process, when posible, is called *confluence*. In this section we describe a general method to transform a Fuchsian DE by confluence into a DE with an irregular singularity.

Suppose we have a Fuchsian DE with singularities at 0, c, and ∞ and that the exponents at $z = c$ and $z = \infty$ depend on c. In order for the DE to have a limiting form as $c \to \infty$, it is necessary to require that the exponents at c and ∞ are linear functions of c. This will be assumed without proof. Thus, we can represent a solution of the DE by the Weierstrass P-function

$$P\begin{pmatrix} 0 & c & \infty & \\ \alpha_{1,1} & \alpha_{1,2} + c\,\beta_{1,2} & \alpha_{1,\infty} + c\,\beta_{1,\infty} & z \\ \alpha_{2,1} & \alpha_{2,2} + c\,\beta_{2,2} & \alpha_{2,\infty} + c\,\beta_{2,\infty} & \end{pmatrix}.$$

From (4.2.4) we get

$$\begin{aligned} y''(z) + &\left[\frac{1-\alpha_{1,1}-\alpha_{2,1}}{z} + \frac{1-\alpha_{1,2}-\alpha_{2,2}-c(\beta_{1,2}+\beta_{2,2})}{z-c}\right] y'(z) + \\ &\left[\frac{\alpha_{1,1}\alpha_{2,1}}{z^2} + \frac{\alpha_{1,2}\alpha_{2,2} + c(\alpha_{1,2}\beta_{2,2}+\alpha_{2,2}\beta_{1,2}) + c^2\beta_{1,2}\beta_{2,2}}{(z-c)^2}\right] y(z) + \\ &\left[\frac{\alpha_{1,\infty}\alpha_{2,\infty} - \alpha_{1,1},\alpha_{2,1} - \alpha_{1,2}\alpha_{2,2}}{z(z-c)}\right] y(z) + \\ &\left[\frac{c(\alpha_{1,\infty}\beta_{2,\infty} + \alpha_{2,\infty}\beta_{1,\infty} - \alpha_{1,2}\beta_{2,2} - \alpha_{2,2}\beta_{1,2}) + c^2(\beta_{1,\infty}\beta_{2,\infty} - \beta_{1,2}\beta_{2,2})}{z(z-c)}\right] y(z) = 0. \end{aligned} \tag{4.5.1}$$

If a limiting form is to exist, we must have $\beta_{1,\infty}\beta_{2,\infty} - \beta_{1,2}\beta_{2,2} = 0$ to avoid the last term in (4.5.1) blowing up. The Fuchsian invariant has value 1, so

$$\alpha_{1,1} + \alpha_{2,1} + \alpha_{1,2} + \alpha_{2,2} + \alpha_{1,\infty} + \alpha_{2,\infty} + c(\beta_{1,2} + \beta_{2,2} + \beta_{1,\infty} + \beta_{2,\infty}) = 1.$$

This equation must hold for all c and for αs and βs independent of c, so

$$\begin{aligned} &\beta_{1,2} + \beta_{2,2} + \beta_{1,\infty} + \beta_{2,\infty} = 0 \\ &\alpha_{1,1} + \alpha_{2,1} + \alpha_{1,2} + \alpha_{2,2} + \alpha_{1,\infty} + \alpha_{2,\infty} = 1. \end{aligned}$$

Now let $c \to \infty$, giving

$$\begin{aligned} y''(z) + &\left[\frac{1-\alpha_{1,1}-\alpha_{2,1}}{z} + \beta_{1,2} + \beta_{2,2}\right] y'(z) + \\ &\left[\frac{\alpha_{1,1}\alpha_{2,1}}{z^2} + \beta_{1,2}\beta_{2,2} - \frac{\alpha_{1,\infty}\beta_{2,\infty} + \alpha_{2,\infty}\beta_{1,\infty} - \alpha_{1,2}\beta_{2,2} - \alpha_{2,2}\beta_{1,2}}{z}\right] y(z) = 0. \end{aligned} \tag{4.5.2}$$

Exercise 4.5.1. *Verify that equation (4.5.1) has an irregular singularity at ∞.*

Example 4.5.1. *Confluence can be used to obtain Bessel's DE of order n. This DE has a regular singular point at $z = 0$ and an irregular singular point at $z = \infty$. The exponents at 0 are $\pm n$, so $\alpha_{1,1} = n$ and $\alpha_{2,1} = -n$, making (4.5.1)*

$$\begin{aligned} y''(z) + &\left[\frac{1}{z} + \beta_{1,2} + \beta_{2,2}\right] y'(z) + \\ &\left[\frac{-n^2}{z^2} + \beta_{1,2}\beta_{2,2} - \frac{\alpha_{1,\infty}\beta_{2,\infty} + \alpha_{2,\infty}\beta_{1,\infty} - \alpha_{1,2}\beta_{2,2} - \alpha_{2,2}\beta_{1,2}}{z}\right] y(z) = 0. \end{aligned}$$

We want to get Bessel's DE,

$$y''(z) + \frac{1}{z}\,y'(z) + \left(1 - \frac{n^2}{z^2}\right)\,y(z) = 0,$$

so, including the condition for the existence of a limiting form and the Fuchsian invariant, we get the following system of equations for the parameters.

$$\begin{gathered}
\beta_{1,2} + \beta_{2,2} = 0 \\
\beta_{1,2}\beta_{2,2} = 1 \\
\alpha_{1,\infty}\beta_{2,\infty} + \alpha_{2,\infty}\beta_{1,\infty} - \alpha_{1,2}\beta_{2,2} - \alpha_{2,2}\beta_{1,2} = 0 \\
\beta_{1,\infty}\beta_{2,infty} - \beta_{1,2}\beta_{2,2} = 0 \\
\beta_{1,2} + \beta_{2,2} + \beta_{1,\infty} + \beta_{2,\infty} = 0 \\
\alpha_{1,2} + \alpha_{2,2} + \alpha_{1,\infty} + \alpha_{2,\infty} = 1
\end{gathered}$$

Since there are six equations in eight unknowns, any solution will contain two undetermined parameters. One such solution is

$$\beta_{1,2} = i, \quad \beta_{2,2} = -i, \quad \beta_{1,\infty} = i, \quad \beta_{2,\infty} = -i, \quad \alpha_{1,\infty} = \tfrac{1}{2} - \alpha_{2,2}, \quad \alpha_{2,\infty} = \tfrac{1}{2} - \alpha_{1,2}.$$

This shows that if we let $c \to \infty$ in the DE defined by

$$P\begin{pmatrix} 0 & c & \infty & \\ n & \alpha_{1,2} + ic & \frac{1}{2} - \alpha_{2,2} + ic & z \\ -n & \alpha_{2,2} - ic & \frac{1}{2} - \alpha_{1,2} - ic & \end{pmatrix},$$

the result is Bessel's DE.

Exercise 4.5.2. *Fill in the details in Example 4.5.1.*

The point of this section is that by means of the process of confluence, known results about the Fuchsian DE can suggest new results or avenues of study for the DE with an irregular singularity. The example involving Bessel's equation is to be taken as an illustration of the process. Actually, much more is known about Bessel's DE than the Fuchsian DE.

Exercise 4.5.3. *Show that the confluent equation obtained by letting $c \to \infty$ in the DE defined by*

$$P\begin{pmatrix} 0 & c & \infty & \\ \frac{1}{2} + m & c - k & -c & z \\ \frac{1}{2} - m & k & 0 & \end{pmatrix}$$

is $\dfrac{d^2u}{dz^2} + \dfrac{du}{dz} + \left(\dfrac{k}{z} + \dfrac{\frac{1}{4} - m^2}{z^2}\right) u = 0$. *Then let $u = e^{-z/2}W_{k,m}(z)$ to get Whittaker's equation for $W_{k,m}$:*

$$\frac{d^2W}{dz^2} + \left[-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right] W = 0.$$

Verify that Whittaker's equation has a regular singular point at 0 and an irregular singular point at ∞.

4.6. Generalized Hypergeometric Functions

A little time spent studying the series form of the basic hypergeometric function,

$$F(\alpha,\beta;\gamma;z)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{z^k}{k!},$$

will suggest the question, "Why be restricted to just α, β, and γ for the factorial functions? Why not allow an arbitrary number of factorials in both numerator and denominator?" (OK, that's two questions, but, as you may know, there are three kinds of mathematicians: those who can count and those who cannot.) Thus we are led to consider the *generalized hypergeometric functions*, denoted ${}_pF_q(\alpha_1,\ldots,\alpha_p;\gamma_1,\ldots,\gamma_q;z)$ and defined by

$${}_pF_q(\alpha_1,\ldots,\alpha_p;\gamma_1,\ldots,\gamma_q;z)=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_k}{\prod_{j=1}^{q}(\gamma_j)_k}\frac{z^k}{k!}.$$

Exercise 4.6.1. *If $p\le q$, the series for ${}_pF_q$ converges for all z.*

Exercise 4.6.2. *If $p=q+1$, the series converges for $|z|<1$ and, unless it terminates, diverges for $|z|\ge 1$.*

Exercise 4.6.3. *If $p>q+1$, the series, unless it terminates, diverges for $z\ne 0$.*

Either p or q or both may be zero, and if this occurs, the absence of parameters will be denoted by a dash, $-$, in the appropriate position.

Example 4.6.1. ${}_0F_0(-;-;z)=\displaystyle\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z.$

Example 4.6.2. ${}_1F_0(\alpha;-;z)=F(\alpha,\beta;\beta;z)=(1-z)^{-\alpha}.$

Example 4.6.3. ${}_0F_1(-;\gamma;z)=\displaystyle\sum_{k=0}^{\infty}\frac{z^k}{(\gamma)_k k!}$, *and from this we get* $J_\nu(z)=\frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1(-;\nu+1;-\frac{z^2}{4})$.

Since the hypergeometric function came from the hypergeometric DE, the generalized hypergeometric functions should also satisfy appropriate DEs. Let the differential operator $\theta=z\frac{d}{dz}$. In terms of θ, the hypergeometric DE is

$$[\theta(\theta+\gamma-1)-z(\theta+\alpha)(\theta+\beta)]\;y=0.$$

Now if

$$y(z)={}_pF_q(z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\gamma_1)_k\cdots(\gamma_q)_k}\frac{z^k}{k!},$$

and since $\theta\,z^k=k\,z^k$, we get

$$\theta\prod_{j=1}^{q}(\theta+\gamma_j-1)\,y=\sum_{k=1}^{\infty}\frac{k\prod_{j=1}^{q}(k+\gamma_j-1)\prod_{i=1}^{p}(\alpha_i)_k}{\prod_{j=1}^{q}(\gamma_j)_k}\frac{z^k}{k!}=\sum_{k=1}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_k}{\prod_{j=1}^{q}(\gamma_j)_{k-1}}\frac{z^k}{(k-1)!}.$$

Shifting the index gives

$$\theta\prod_{j=1}^{q}(\theta+\gamma_j-1)\,y=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_{k+1}}{\prod_{j=1}^{q}(\gamma_j)_k}\frac{z^{k+1}}{k!}=z\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i+k)\prod_{i=1}^{p}(\alpha_i)_k}{\prod_{j=1}^{q}(\gamma_j)_k}\frac{z^k}{k!}=z\prod_{i=1}^{p}(\theta+\alpha_i)\,y.$$

Thus, if $p \le q+1$, we see that $y = {}_pF_q$ satisfies

$$\left[\theta \prod_{j=1}^{q} (\theta + \gamma_j - 1) - z \prod_{i=1}^{p} (\theta + \alpha_i)\right] y = 0.$$

This DE is of order $q+1$ and, if no γ_j is a positive integer and no two γ_j's differ by an integer, the general solution is

$$y = \sum_{m=0}^{q} c_m\, y_m$$

where, for $m = 1, 2, \ldots, q$,

$$\begin{aligned} y_0 &= {}_pF_q(\alpha_1, \ldots, \alpha_p; \gamma_1, \ldots, \gamma_q; z) \\ y_m &= z^{1-\gamma_m}\, {}_pF_q(\alpha_1 - \gamma_m + 1, \ldots, \alpha_p - \gamma_m + 1; \\ &\quad \gamma_1 - \gamma_m + 1, \ldots, \gamma_{m-1} - \gamma_m + 1, 2 - \gamma_m, \gamma_{m+1} - \gamma_m + 1, \ldots, \gamma_q - \gamma_m + 1; z) \end{aligned}$$

Exercise 4.6.4. *Find the DE satisfied by ${}_3F_2(2, 2, 2; \frac{5}{2}, 4; z)$. Also find the general solution of this DE.*

Exercise 4.6.5. *Show that if y_1 and y_2 are linearly independent solutions of*

$$y''(z) + p(z)\, y'(z) + q(z)\, y(z) = 0$$

then three linearly independent solutions of

$$w'''(z) + 3p(z)\, w''(z) + \left(2p^2(z) + p'(z) + 4q(z)\right)\, w'(z) + \left(4p(z)q(z) + 2q'(z)\right)\, w(z) = 0$$

are $y_1^2(z)$, $y_1(z)y_2(z)$, and $y_2^2(z)$.

Exercise 4.6.6. *Show that ${}_3F_2(2, 2, 2; \frac{5}{2}, 4; z) = \left(F(1, 1; \frac{5}{2}; z)\right)^2$. [Hint: Use Exercises 4.6.4 and 4.6.5.]*