Special Functions

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Chapter 5. Orthogonal Functions

5.1. Generating Functions

Consider a function f of two variables, (x, t), and its formal power series expansion in the variable t:

$$f(x,t) = \sum_{k=0}^{\infty} g_k(x)\, t^k.$$

The coefficients in this series are, in general, functions of x, and we can think of them as having been "generated" by the function f. In fact, $g_k(x) = \dfrac{1}{k!}\dfrac{\partial^k f}{\partial t^k}(x,0)$, though there may be better ways to compute them. If this idea is extended slightly, we get the following definition:

Definition 5.1.1. *The function $F(x,t)$ is a generating function for the sequence $\{g_k(x)\}$ if there exists a sequence of constants $\{c_k\}$ such that*

$$F(x,t) = \sum_{k=0}^{\infty} c_k g_k(x)\, t^k.$$

It is not uncommon for all the c_ks to be one. One of the principal problems involving generating functions is determining a generating function for a given set or sequence of polynomials. Especially desirable is a general theory which can be used to get generating functions. Unfortunately, no such theory has yet been developed, so we must be content with results for special cases found using manipulative dexterity.

Example 5.1.1. *One special case is when the coefficients are successive powers of the same function. Let $\{g_k(x)\} = \{(f(x))^k\}$. Then the generating function can be found using the formula for the sum of a geometric series.*

$$F(x,t) = \sum_{k=0}^{\infty} (f(x))^k\, t^k = \frac{1}{1 - t\, f(x)}$$

provided $|f(x)| < 1$.

Exercise 5.1.1. *Find the generating function for the sequence $\{k\,(f(x))^k\}$.*

Many sets of elementary and special functions have known generating functions. Here are some examples.

Example 5.1.2. *The Bernoulli functions. Let $y(x,t) = \sum_{k=0}^{\infty} B_k(x)\, t^k$. Termwise differentiation with respect to x and properties of the Bernoulli functions (section 1.3) yields $y_x(x,t) = t\, y(x,t)$. Thus,*

$$\frac{\partial}{\partial x}\left[e^{-xt} y(x,t)\right] = e^{-xt}\left[y_x(x,t) - t\, y(x,t)\right] = 0,$$

and so for each t there is a function $C(t)$ such that $y(x,t) = C(t)e^{xt}$, and we have

$$C(t)e^{xt} = \sum_{k=0}^{\infty} B_k(x)\, t^k. \tag{5.1.1}$$

Integration of (5.1.1) over the interval $[0,1]$ and properties of the Bernoulli functions give

$$C(t) = \frac{t}{e^t - 1}.$$

Thus, the generating function for the Bernoulli functions is $y(x,t) = \dfrac{te^{xt}}{e^t - 1}$.

Exercise 5.1.2. *Fill in the details in Example 5.1.2.*

Example 5.1.3. *Legendre polynomials:* $(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} P_k(x)t^k$.

Exercise 5.1.3. *Use Taylor's theorem to verify the first three coefficients in the generating function relation for the Legendre polynomials.*

Example 5.1.4. *Bessel functions:* $\exp\left[\frac{1}{2}z(t - \frac{1}{t})\right] = \sum_{k=-\infty}^{\infty} J_k(z)t^k$.

Example 5.1.5. *Hermite polynomials. Denote the Hermite polynomial of degree* n *by* $H_n(x)$. *Then* $\exp\left(2xt - t^2\right) = \sum_{n=0}^{\infty} \dfrac{H_n(x)}{n!}\, t^n$.

Exercise 5.1.4. *Find the first four Hermite polynomials.*

Exercise 5.1.5. *Prove the expansions*

$$e^{t^2} \cos 2xt = \sum_{n=0}^{\infty} \frac{(-1)^n H_{2n}(x)}{(2n)!}\, t^{2n}$$
$$e^{t^2} \sin 2xt = \sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}(x)}{(2n+1)!}\, t^{2n+1}$$

for $|t| < \infty$. *These expressions can be thought of as generating functions for the even and odd Hermite polynomials.*

The generating functions for both the Legendre and the Hermite polynomials are functions of the form $G(2xt - t^2)$. The following theorem is representative of theorems which give properties common to all sets of functions having generating functions of this form.

Theorem 5.1.1. *If* $G(2xt - t^2) = \sum_{n=0}^{\infty} g_n(x)\, t^n$, *then* $g_0'(x) = 0$ *and, for* $n \geq 1$, *the* g_n*s satisfy the differential-difference equation*

$$x\, g_n'(x) - n\, g_n(x) = g_{n-1}'(x).$$

Proof. Let $F(x,t) = G(2xt - t^2) = \sum_{n=0}^{\infty} g_n(x)\, t^n$. Then F satisfies the PDE $(x - t)\dfrac{\partial F}{\partial x} - t\dfrac{\partial F}{\partial t} = 0$. In terms of the series, this PDE is

$$\sum_{n=0}^{\infty} x\, g_n'(x)\, t^n - \sum_{n=0}^{\infty} n\, g_n(x)\, t^n = \sum_{n=1}^{\infty} g_{n-1}'(x)\, t^n.$$

Equating coefficients gives the desired result. ♠

Exercise 5.1.6. *In A&S, pages 783-4, a number of generating functions are given as functions of* $R = \sqrt{1 - 2xt + t^2}$. *Formulate and prove the equivalent of Theorem 5.1.1 using* R *in place of* $2xt - t^2$.

5.2. Orthogonality

Consider the DE

$$a_0(x)\, y'' + a_1(x)\, y' + [a_2(x) + \lambda]\, y = 0.$$

Multiply by the "integrtating factor" $p(x) = \exp \frac{a_1(x)}{a_0(x)}\, dx$, let $q(x) = \frac{a_2(x)}{a_0(x)}\, p(x)$, and $r(x) = \frac{p(x)}{a_0(x)}$, to get the DE into the form

$$[p(x)\, y']' + [q(x) + \lambda\, r(x)]\; y = 0. \tag{5.2.1}$$

Equation (5.2.1) is said to be in *Sturm-Liouville form* and if appropriate boundary conditions are specified on an interval we have a *Sturm-Liouville problem.* Values of λ for which a Sturm-Liouville problem (SLP) has nontrivial solutions are called *eigenvalues* of the SLP and the corresponding solutions are called *eigenfunctions.* These ideas are studied in detail in courses on partial differential equations and boundary value problems where the SLP arises naturally in the solution of PDEs with boundary conditions. The following theorem, stated here rather vaguely, is proved in such courses.

Theorem 5.2.1. *Under appropriate conditions, if* y_m *and* y_n *are eigenfunctions corresponding to distinct eigenvalues of the SLP associated with (5.2.1) on the interval* $[a, b]$, *then*

$$\int_a^b r(x) y_m(x) y_n(x)\, dx = 0. \tag{5.2.2}$$

When equation (5.2.2) holds, we say that y_m and y_n are *orthogonal* with respect to the weight function $r(x)$. This equation can, in fact, be taken as the definition of orthogonality.

Exercise 5.2.1. *What do you call a tornado at the Kentucky Derby?*

Example 5.2.1 (Legendre). *Legendre's DE, as we have seen, is* $(1 - x^2)\, y'' - 2\, x\, y' + n(n + 1)\, y = 0$. *In Sturm-Liouville form, this becomes*

$$\left[(1 - x^2)\, y'\right]' + n(n + 1)\, y = 0.$$

Here, $p(x) = 1{-}x^2$, $q(x) \equiv 0,$, $r(x) \equiv 1$, *and* $\lambda = n(n{+}1)$. *Since* $x = \pm 1$ *are regular singular points, we can be sure solutions exist on the closed interval* $[-1, 1]$ *only when the solutions are polynomials, so the eigenvalues are* $n = 0, 1, 2, \ldots$ *and the eigenfunctions are the corresponding Legendre polynomials* $P_0(x), P_1(x), P_2(x), \ldots$. *By Theorem 5.2.1 we have*

$$\int_{-1}^1 P_m(x)\, P_n(x)\, dx = 0$$

whenever m *and* n *are distinct nonnegative integers.*

The orthogonality integral is a generalization to functions of the dot product for vectors, and since the dot product of a vector with itself is the square of the length of the vector, the integral in (5.2.2) with both eigenfunctions the same can be interpreted as the "length" squared of the eigenfunction. Often, we want this length to be one for all the eigenfunctions, in which case we say that the eigenfunctions are *normalized.* Since the eigenfunctions are orthogonal by (5.2.2), if they are also normalized, we say they are *orthonormal.*

Example 5.2.1 (continued). *We now determine* $\int_{-1}^{1} P_n^2(x)\,dx$ *using the generating function.*

$$
\begin{aligned}
(1-2xt+t^2)^{-\frac{1}{2}} &= \sum_{n=0}^{\infty} P_n(x)\,t^n \\
(1-2xt+t^2)^{-1} &= \left[\sum_{n=0}^{\infty} P_n(x)\,t^n\right]^2 \\
\frac{1}{t}\log\left|\frac{1+t}{1-t}\right| &= \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1} P_n^2(x)\,dx \\
2\left(1+\frac{t^2}{3}+\frac{t^4}{5}+\cdots+\frac{t^{2n}}{2n+1}+\cdots\right) &= \sum_{n=0}^{\infty} t^{2n}\int_{-1}^{1} P_n^2(x)\,dx.
\end{aligned}
$$

Equating coefficients gives the normalizing constants for the Legendre polynomials:

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}.$$

Exercise 5.2.2. *Fill in the details in Example 5.2.1.*

Example 5.2.2 (Bessel). *Bessel's DE of order* n*, (slightly modified - do you see how?)* $x^2y'' + x\,y' + (\lambda^2x^2 - n^2)\,y = 0$*, written in Sturm-Liouville form is*

$$[x\,y']' + \left[\lambda^2 x - \frac{n^2}{x}\right]\,y = 0.$$

For the interval $[0, b]$*, the eigenvalues are* $\lambda_k = \frac{\alpha_k}{b}$*, where* α_k *is the* k^{th} *positive zero of* $J_n(x)$*. The orthogonality integral is, for* $m \neq k$*,*

$$\int_0^b x\,J_n(\lambda_m x)\,J_n(\lambda_k x)\,dx = 0.$$

Note here that n *is fixed, and the different eigenvalues and eigenfunctions are denoted by the subscripts on* λ *or* α*.*

For sets of polynomials, the following equivalent condition for orthogonality is often useful. We call a set of polynomials *simple* if the set contains exactly one polynomial of each degree; unless stated otherwise, the degree of a subscripted polynomial is equal to its subscript.

Theorem 5.2.2. *If* $\{\phi_n(x)\}$ *is a simple set of real polynomials and* $r(x) > 0$ *on an interval* (a, b)*, then* $\{\phi_n(x)\}$ *is an orthogonal set with respect to the weight function* $r(x)$ *if and only if for* $k = 0, 1, 2, \ldots, n-1$*,*

$$\int_a^b r(x)\,x^k\,\phi_n(x)\,dx = 0.$$

Outline of Proof. The proof is based first on the fact that any polynomial of degree $m < n$ can be written as a linear combination of powers of x from x^0 through x^m. Then the fact that x^k can be expressed as a linear combination of $\phi_0(x)$ through $\phi_k(x)$ is used. Details are left to the student. ♠

Exercise 5.2.3. *Prove Theorem 5.2.2.*

Exercise 5.2.4. *Prove that if* $\{\phi_n(x)\}$ *is a simple set of real polynomials and* $r(x) > 0$ *on an interval* (a, b)*, then for every polynomial* P *of degree less than* n*,* $\int_a^b r(x)\,\phi_n(x)\,P(x)\,dx = 0$*. Also prove that* $\int_a^b r(x)\,x^n\,\phi_n(x)\,dx \neq 0$.

The interesting part of a polynomial is near the zeros. After the last zero and before the first one, polynomials are rather boring - they either go up, up, up, or down, down, down.

Theorem 5.2.3. *If* $\{\phi_n(x)\}$ *is a simple set of real polynomials, orthogonal with respect to a weight function* $r(x) > 0$ *on an interval* (a, b)*, then, for each* n*, the zeros of* ϕ_n *are distinct and all lie in the interval* (a, b)*.*

Proof. For $n > 0$, by Theorem 5.2.2 $\int_a^b r(x)\,\phi_n(x)\,dx = 0$, so the integrand must change sign at least once in (a, b), and since $r(x) > 0$, this means $\phi_n(x)$ changes sign in (a, b). Let $\{\alpha_k\}_{k=1}^{s}$ be the set of points where $\phi_n(x)$ changes sign in (a, b). These are the zeros of ϕ_n of odd multiplicity, and since the degree of ϕ_n is n, we know that $s \leq n$. Form the polynomial

$$P(x) = \prod_{k=1}^{s}(x - \alpha_k).$$

Assume $s < n$. Then by Exercise 5.2.4,

$$\int_a^b r(x)\,\phi_n(x)\,P(x)\,dx = 0.$$

But all the zeros of $\phi_n(x)\,P(x)$ are of even multiplicity, so $r(x)\,\phi_n(x)\,P(x)$ cannot change sign in (a, b). Hence, $s < n$ is not possible, and we must have $s = n$. This means that ϕ_n has n roots of odd multiplicity in (a, b). Since the degree of ϕ_n is n, each root is simple, and the theorem is proved. ♠

5.3. Series Expansions

An important application of orthogonal polynomials in physics and engineering is the expansion of a given function in a series of the polynomials. For a simple set of polynomials, the powers of x in the usual series representation are replaced by the polynomials of appropriate degree. Of course, the problem is to find the coefficients in such a series expansion, and this is where orthogonality becomes quite useful.

Example 5.3.1. *Let* f *be defined in the interval* $(-1, 1)$*, and expand* $f(x)$ *in a series of Legendre polynomials. In other words, we want to determine the coefficients in*

$$f(x) = \sum_{n=0}^{\infty} c_n\,P_n(x) \tag{5.3.1}$$

so that equality holds for $x \in (-1, 1)$*. Proceeding formally, we multiply both sides by* $P_m(x)$ *and integrate from* -1 *to* 1*.*

$$\int_{-1}^{1} f(x)\,P_m(x)\,dx = \sum_{n=0}^{\infty} c_n \int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \frac{2}{2m+1}c_m,$$

which implies that, for $n = 0, 1, 2, \ldots$

$$c_n = (n + \frac{1}{2}) \int_{-1}^{1} f(x)\, P_n(x)\, dx. \qquad (5.3.2)$$

This procedure is neat, clean, and algorithmic, but we took some mathematical liberties which should at least be acknowledged. In particular, how did we know that $f(x)$ could be represented as in (5.3.1) in the first place, and also, was it legitimate to interchange the operations of integration and summation? Unless these points are cleared up, we have no guarantee, except faith, that (5.3.1) with coefficients given by (5.3.2) converges and has sum $f(x)$. Another concern is that even if we can be sure the procedure works for Legendre polynomials, will a similar procedure be valid for a different set of simple orthogonal polynomials? Fortunately, for a given set of orthogonal polynomials, there are conditions which do guarantee that equations (5.3.1) and (5.3.2) or their equivalents are valid. Unfortunately, the conditions are different for different sets of polynomials. Proofs get somewhat involved, and are omitted here, but interested readers may consult Lebedev or Whittaker and Watson.

Theorem 5.3.1. *If the real function f is piecewise smooth in the interval $(-1, 1)$ and if $\int_{-1}^{1} f^2(x)\, dx$ is finite, then the Legendre series (5.3.1) with coefficients given by (5.3.2) converges to $f(x)$ wherever f is continuous. If x_0 is a point of discontinuity, the series converges to the average of the right-hand and left-hand limits of $f(x)$ at x_0.*

Exercise 5.3.1. *Expand $f(x) = x^2$ in a series of Legendre polynomials.*

Exercise 5.3.2. *Expand $f(x) = \begin{cases} 0, & -1 \le x < \alpha \\ 1, & \alpha < x \le 1 \end{cases}$ in a series of Legendre polynomials, and verify the value at $x = \alpha$.*

Exercise 5.3.3. *Express $f(x) = \sqrt{\frac{1-x}{2}}$ in a series of Legendre polynomials. Calculate the coefficients by using the generating function.*

It is possible to derive all properties of a set of orthogonal polynomials by starting with only the generating function. The following series of exercises builds up some results about the Hermite polynomials defined in Example 5.1.5.

Exercise 5.3.4. *Show that the generating function $F(x, t)$ for the Hermite polynomials satisfies $\frac{\partial F}{\partial x} - 2t\, F = 0$, and so $H_n'(x) = 2n\, H_{n-1}(x)$. Similarly, show that $F(x, t)$ satisfies $\frac{\partial F}{\partial t} - 2(x - t)\, F = 0$, and so*

$$H_{n+1}(x) - 2x\, H_n(x) + 2n\, H_{n-1}(x) = 0. \qquad (5.3.3)$$

Exercise 5.3.5. *Show that the Hermite polynomials satisfy the differential equation (Hermite's DE)*

$$y''(x) - 2x\, y'(x) + 2n\, y(x) = 0.$$

Write Hermite's DE in Sturm-Liouville form and determine the interval and the weight function for the orthogonality of the Hermite polynomials.

Exercise 5.3.6. *In this exercise, you will calculate $\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)\, dx$. Begin by replacing the index n in (5.3.3) by $n - 1$ and multiply by $H_n(x)$. Then from this equation subtract (5.3.3) multiplied by $H_{n-1}(x)$.*

Work with this result to obtain

$$\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)\,dx = 2n \int_{-\infty}^{\infty} e^{-x^2} H_{n-1}^2(x)\,dx$$

for $n = 2, 3, \ldots$. *Repeated application of this reduction formula gives, for* $n = 2, 3, \ldots$,

$$\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)\,dx = 2^n\, n!\, \sqrt{\pi}. \tag{5.3.4}$$

Finally, show by direct calculation that (5.3.4) also holds for $n = 0, 1$.

There is a result for Hermite polynomials corresponding to Theorem 5.3.1, in which the integral required to be finite is $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} f^2(x)\,dx$.

Exercise 5.3.7. *Expand* $f(x) = sgn(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$ *in a series of Hermite polynomials.*

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