Analytical Dynamics: Lagrange's Equation and its Application – A Brief Introduction

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Contents

1 The Calculus of Variations

The Euler-Lagrange formulation was built upon the foundation of the the calculus of variations, the initial development of which is usually credited to Leonhard Euler.[1] The calculus of variations is an extensive subject, and there are many fine references which present a detailed development of the subject – *see*

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[1] http://en.wikipedia.org/wiki/Calculus_of_variations

1

Bibliography. The purpose of this addendum is do provide a brief background in the theory behind Lagrange's Equations. Fortunately, complete understanding of this theory is not *absolutely necessary* to use Lagrange's equations, but a *basic* understanding of variational principles can greatly increase your mechanical modeling skills.

1.1   Extremum of an Integral – The Euler-Lagrange Equation

Given the Integral of a functional (a function of functions) of the form

$$I(\epsilon) = \int_{t_1}^{t_2} F(U, \dot{U}, t)dt, \tag{1}$$

where t_1, and t_2 are arbitrary, ϵ is a small real and independent variable, and U and $\dot{U}$ are given by

$$U(t) = u(t) + \epsilon\eta(t), \text{ and } \dot{U}(t) = \dot{u}(t) + \epsilon\dot{\eta}(t). \tag{2}$$

The functions U, and u may be thought of as describing the possible positions of a dynamical system between the two instants in time, t_1, and t_2, where $u(t)$ represents the position when the integral described by Equation (1) is stationary, i.e. where it is an extremum, and $U(t)$ is $u(t)$ plus a *variation* $\epsilon\eta(t)$. The function $U(t)$ does not by definition render (1) stationary because we shall assume $\eta(t)$ is *independent* of $u(t)$, and we will assume that a *unique* function renders (1) an extremum. The reasons for these assumptions will become clear below. The important point so far is that have not made any restrictive statements about $I(\epsilon)$ other than it is an integral of a functional of the functions $U(t)$ and $\dot{U}(t)$. We will now specify that the functions $u(t)$, and $\eta(t)$ are of class C^2. That is, they possess continuous second derivatives with respect to t. Further, let us stipulate that $\eta(t)$ must vanish at $t = t_1$, and $t = t_2$. In other words, $u(t)$ and $U(t)$ coincide at the end points of the interval $[\,t_1,\,t_2]$, where t_1, and t_2 are *arbitrary.*

Now that we have the stage more or less set up, lets see what rules the functional $F(U, \dot{U}, t)$ must obey to render (1) extreme. We have, by definition, that the function $u(t)$ renders I stationary, hence, we know this occurs when $U(t) = u(t)$, or $\epsilon = 0$. this situation is depicted in Figure 1.
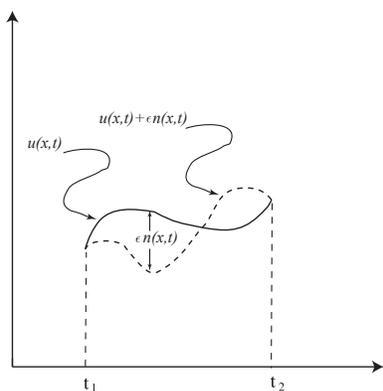
Figure 1. Relationship between extremizing function $u(t)$, and variation $\epsilon\eta(t)$.

Thus, assuming that t_1, and t_2 are not functions of ϵ, we set the first derivative of $I(\epsilon)$ equal to zero.

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{t_1}^{t_2} \frac{dF}{d\epsilon}(U, \dot{U}, t)dt = 0. \tag{3}$$

However,

$$\frac{dF}{d\epsilon}(U, \dot{U}, t) = \frac{\partial F}{\partial U}\frac{\partial U}{\partial \epsilon} + \frac{\partial F}{\partial \dot{U}}\frac{\partial \dot{U}}{\partial \epsilon}, \tag{4}$$

so substituting Equation (4) into Equation (3), and setting $\epsilon = 0$, we have

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{t_1}^{t_2}\left(\frac{\partial F}{\partial u}\eta + \frac{\partial F}{\partial \dot{u}}\dot{\eta}\right)dt = 0. \tag{5}$$

Integration of Equation (5) by parts yields:

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{t_1}^{t_2}\eta(t)\left(\frac{\partial F}{\partial u} - \frac{d}{dt}\frac{\partial F}{\partial \dot{u}}\right)dt + \left.\frac{\partial F}{\partial \dot{u}}\eta(t)\right|_{t_1}^{t_2} = 0. \tag{6}$$

The last term in Equation (6) vanishes because of the stipulation $\eta(t_1) = \eta(t_2) = 0$, which leaves

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \int_{t_1}^{t_2}\eta(t)\left(\frac{\partial F}{\partial u} - \frac{d}{dt}\frac{\partial F}{\partial \dot{u}}\right)dt = 0. \tag{7}$$

By the fundamental theorem of the calculus of variations [1], since $\eta(t)$ is arbitrary except at the end points t_1, and t_2, we must have, in general

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \frac{\partial F}{\partial u} - \frac{d}{dt}\frac{\partial F}{\partial \dot{u}} = 0. \tag{8}$$

Equation (8) is known as the Euler-Lagrange equation. It specifies the conditions on the functional F to extremize the integral $I(\epsilon)$ given by Equation (1). By extremize, we mean that $I(\epsilon)$ may be (1) maximum, (2) minimum, or (3) an inflection point – i.e. neither maximum, nor minimum. In fact, there is no guarantee of the existence of a *global* extremum; the integral may be only locally extreme for *small* values of ϵ. The determination of the nature of the stationary condition of $I(\epsilon)$ for the general case is beyond the scope of this document. Let it suffice to say that for *every* case considered in the study of mechanics or dynamics, $I(\epsilon)$ will be *globally minimum* when $\epsilon = 0$.

Equation (5) is often written

$$\delta I = \left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = \epsilon\int_{t_1}^{t_2}\left(\frac{\partial F}{\partial u}\eta + \frac{\partial F}{\partial \dot{u}}\dot{\eta}\right)dt = \int_{t_1}^{t_2}\left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial \dot{u}}\delta\dot{u}\right)dt = 0, \tag{9}$$

where $\delta u = \eta$ is the *variation* of u, and [2],

$$\frac{d}{dt}\delta u = \frac{d}{dt}(\epsilon\eta) = \epsilon\dot{\eta} = \delta\frac{du}{dt} = \delta\dot{u}. \tag{10}$$

Using Equation (10), and integrating Equation (9) by parts, we obtain

$$\delta I = \int_{t_1}^{t_2}\left(\frac{\partial F}{\partial u} - \frac{d}{dt}\frac{\partial F}{\partial \dot{u}}\right)\delta u \, dt, \tag{11}$$

with the stipulation, as before, that $\delta u(t_1) = \delta u(t_2) = 0$.

2   Hamilton's Principle

Hamilton's principal is, perhaps, the most important result in the calculus of variations. We derived the Euler-Lagrange equation for a single variable, u, but we will now shift our attention to a system N particles of mass m_i each. Hence, we may obtain N equations of the form

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i, \tag{12}$$

where the **bold** font indicates a *vector* quantity, and $\mathbf{F}_i$ denotes the *total* force on the ith particle. D'Alembert's principle may be stated by rewriting Equation (12) as

$$m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i = 0 \tag{13}$$

Taking the *dot product* of each of the Equations (13) with the variation in position $\delta \mathbf{r}$, and summing the result over all N particles, yields

$$\sum_{i=1}^{N} \left(m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i \right) \cdot \delta \mathbf{r}_i = 0. \tag{14}$$

We note that the sum of the *virtual work* done by the applied forces over the virtual displacements is given by

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i. \tag{15}$$

Next, we note that

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_{i=1}^{N} m_i \left[\frac{d}{dt} \left(\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \right) - \delta \left(\frac{1}{2} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \right) \right] = \sum_{i=1}^{N} m_i \frac{d}{dt} \left(\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \right) - \delta T, \tag{16}$$

where δT is the variation of the kinetic energy. Hence, Equation (16) may be written,

$$\delta T + \delta W = \sum_{i=1}^{N} m_i \frac{d}{dt} \left(\dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \right). \tag{17}$$

In a manner similar to that shown in Figure 1, and in view of Equation (10) the possible dynamical paths of each particle may be represented as shown in Figure 2, where the *varied* dynamical path may be thought to occur *atemporally*.
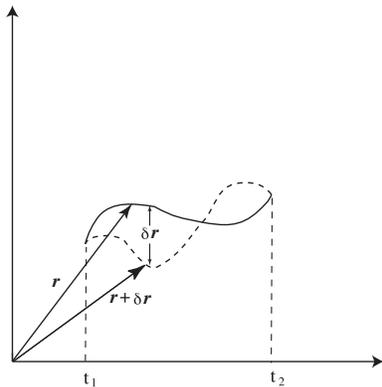
Figure 2. Possible dynamical paths for a particle between two arbitrary instants in time.

Since we again have that $\delta\mathbf{r}(t_1) = \delta\mathbf{r}(t_2) = 0$, we may multiply Equation (17) by dt, and and integrate between the two arbitrary times t_1, and t_2 to obtain

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = \sum_{i=1}^{N} m_i\, (\dot{\mathbf{r}}_i \cdot \delta\mathbf{r}_i)\Big|_{t_1}^{t_2} = 0. \tag{18}$$

If δW can be expressed as the variation of the potential energy, $-\delta V^2$, Equation (18) may be written

$$\int_{t_1}^{t_2} (\delta T - \delta V)\, dt = 0. \tag{19}$$

Introducing the Lagrange function, $L = T - V$, Equation (19) becomes

$$\delta \int_{t_1}^{t_2} L dt = 0. \tag{20}$$

Equation (20) is the mathematical statement of *Hamilton's principal.* Hamilton's principal may be defined in words as follows.

Definition 2.1 *The actual path a body takes in configuration space renders the value of the definite integral $I = \int_{t_1}^{t_2} L dt$ stationary with respect to all arbitrary variations of the path between two instants t_1, and t_2 provided that the path variations vanish at t_1, and t_2 [2]. The integral defined in Equation (20) is usually referred to as the* action integral.[3]

In all physical systems, the stationary value of I will be a minimum.

2.1   Generalized Coordinates and Forces

Implicit in the definition of Hamilton's principle is that the system will move along a dynamical path consistent with the system constraints – i.e. along a *permissible* path. *Generalized coordinates* render the dynamical path explicitly permissible by describing it using the *minimum* number of independent coordinates. Thus, the ith system position may be described as a function of the N generalized coordinates, and (in general) time, t, as follows:

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, q_3, \cdots, q_n, t). \tag{21}$$

Hence, the variation of the ith position, which occurs atemporally, may be expressed as

$$\delta\mathbf{r}_i = \sum_{j=1}^{N} \frac{\partial\mathbf{r}_i}{\partial q_j} \delta q_j. \tag{22}$$

Generalized forces forces are those forces which do work (or virtual work) through displacement of the generalized coordinates. Thus, the virtual work done by an applied force is given by

$$\delta W_i = \mathbf{F}_i \cdot \delta\mathbf{r}_i = \mathbf{F}_i \cdot \sum_{j=1}^{N} \frac{\partial\mathbf{r}_i}{\partial q_j} \delta q_j = \sum_{j=1}^{N} \mathbf{F}_i \cdot \frac{\partial\mathbf{r}_i}{\partial q_j} \delta q_j = \sum_{j=1}^{N} Q_{q_j} \delta q_j, \tag{23}$$

where

$$Q_{q_j} = \mathbf{F}_i \cdot \frac{\partial\mathbf{r}_i}{\partial q_j}. \tag{24}$$

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[2]The negative sign on δV is chosen to reflect that conservative forces may always be written as the negative gradient of the potential energy: $\mathbf{F}_{ci} = -\boldsymbol{\nabla} V$. Where, for example, in Cartesian coordinates: $\boldsymbol{\nabla} = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$.

[3]See: https://en.wikipedia.org/wiki/Action_(physics).

In general, the number of number of position vectors locating the center of mass of the bodies in a system is always equal to the number of bodies, and the number of degrees of freedom is always equal to the number of required generalized coordinates. Hence, letting M denote the number of bodies in a system, and N the number of degrees of freedom, the total virtual work done on a system of M bodies over N degrees of freedom is given by

$$\delta W = \sum_{i=1}^{M} \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_{i=1}^{M} \mathbf{F}_i \cdot \sum_{j=1}^{N} \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_{i=1}^{M} \sum_{j=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_{j=1}^{N} Q_{q_j}^i \delta q_j \qquad (25)$$

where

$$Q_{q_j}^i = \sum_{i=1}^{M} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}. \qquad (26)$$

Similarly, the virtual work done by the the ith resultant torque on a rotating body described by N_R generalized rotational coordinates is given by

$$\delta W_i = \boldsymbol{\tau}_i \cdot \sum_{j=1}^{N_R} \frac{\partial \boldsymbol{\theta}_i}{\partial \phi_j} \delta \phi_j \qquad (27)$$

where we identify the rotational coordinate $\boldsymbol{\theta}_i$ as a function of N_R generalized rotational coordinates:

$$\boldsymbol{\theta}_i = \boldsymbol{\theta}_i(\phi_1, \phi_2, \cdots, \phi_{N_R}). \qquad (28)$$

Hence the total virtual work done by torques may be expressed as

$$\delta W_\theta = \sum_{i=1}^{M_I} \boldsymbol{\tau}_i \cdot \sum_{k=1}^{N_R} \frac{\partial \boldsymbol{\theta}_i}{\partial \phi_k} \delta \phi_k = \sum_{i=1}^{M_I} \sum_{k=1}^{N_R} \boldsymbol{\tau}_i \cdot \frac{\partial \boldsymbol{\theta}_i}{\partial \phi_k} \delta \phi_k = \sum_{k=1}^{N_R} Q_{\phi_k}^i \delta \phi_k, \qquad (29)$$

where M_I denotes the number of inertias acted upon by resultant torques in a system. Thus, to account for rotational degrees of freedom, Equation (25) must be modified adding the work done by applied torques:

$$\delta W = \sum_{j=1}^{N} Q_{q_j}^i \delta q_j + \sum_{k=1}^{N_R} Q_{\phi_k}^i \delta \phi_k \qquad (30)$$

where

$$Q_{\phi_k}^i = \sum_{i=1}^{M_I} \boldsymbol{\tau}_i \cdot \frac{\partial \boldsymbol{\theta}_i}{\partial \phi_k}. \qquad (31)$$

Consider the following example.

Example 2.1 *Given the system shown in Figure 3, determine the virtual work δW done by the force $\mathbf{F} = 4\mathbf{i} + 3\mathbf{j}$, where $\mathbf{i}$, and $\mathbf{j}$ are the unit vectors in the x and y directions, over a virtual displacement $\delta \mathbf{r}$ consistent with the constraints: $x = r \cos \theta$, and $y = r \sin \theta$. Use the generalized coordinates, $q_1 = r$, and $q_2 = \theta$.*

Solution:

This system consists of a single mass, so $M = 1$, and two degrees of freedom, so $N = 2$. We have a choice to use either the x, y-coordinate system, in which the force $\mathbf{F}$ is described, or the r, θ-coordinate system, which trivially describes the motion of the particle in terms of the unit vectors, $\mathbf{e}_r$, and $\mathbf{e}_\theta$. Let
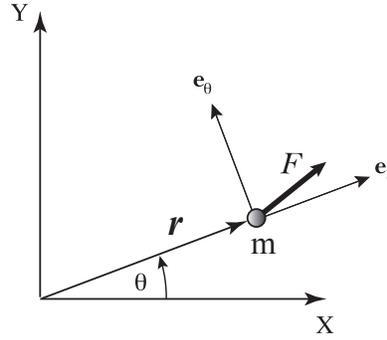
Figure 3. Particle of mass m moving under influence of force $\mathbf{F}$.

us choose the former choice to better illustrate the use of generalized coordinates. Writing the location of the mass in terms of x, and y, we have

$$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}. \tag{32}$$

Applying Equation (22), to Equation (32), we obtain

$$\delta\mathbf{r} = \sum_{j=1}^{N} \frac{\partial\mathbf{r}}{\partial q_j}\delta q_j = (\cos\theta\mathbf{i} + \sin\theta\mathbf{j})\,\delta r + (-r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j})\,\delta\theta. \tag{33}$$

The virtual work is given by $\delta W = \mathbf{F} \cdot \delta\mathbf{r}$, hence, we obtain

$$\delta W = (4\mathbf{i} + 3\mathbf{j}) \cdot [(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})\,\delta r + (-r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j})\,\delta\theta], \tag{34}$$

or,

$$\delta W = 4\,(\cos\theta\delta r - r\sin\theta\delta\theta) + 3\,(\sin\theta\delta r + r\cos\theta\delta\theta). \tag{35}$$

or

$$\delta W = (4\cos\theta + 3\sin\theta)\delta r + (3\cos\theta - 4\sin\theta)r\delta\theta \tag{36}$$

Careful examination of Equation (36) reveals that the coefficients of the δr, and $r\delta\theta$ terms represent the the components of the applied force $\mathbf{F}$ in the $\mathbf{e}_r$, and $\mathbf{e}_\theta$ directions respectively. Hence the generalized forces in this case are defined as

$$Q_R = 4\cos\theta + 3\sin\theta \tag{37}$$

and

$$Q_\theta = 3\cos\theta - 4\sin\theta \tag{38}$$
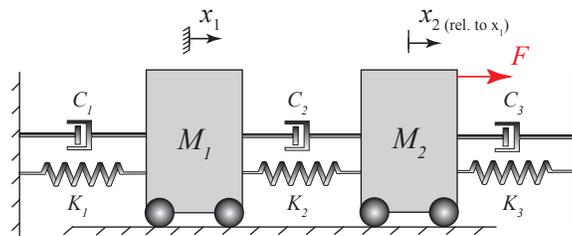
Figure 4. Forced two mass system with a relative coordinate.

Example 2.2 *Consider the system shown in Figure (4) which consists of two viscoelastically connected masses described using one absolute and one relative coordinate.*

1. *Determine the displacement of the masses using the generalized coordinates, $q_1 = x_1$ and $q_2 = x_2$, and the resulting virtual displacements.*

2. *Identify the applied forces.*

3. *Determine the generalized forces, Q_{x_1} and Q_{x_2}.*

Solution:

(a) The displacement of the first mass, M_1, is given by $\mathbf{r}_1 = x_1\mathbf{i}$. Since the displacement of the second mass is described relative to the first, we have $\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{r}_{r_2/r_1} = (x_1 + x_2)\mathbf{i}$. Hence, the virtual displacements are $\delta\mathbf{r}_1 = \delta x_1\mathbf{i}$, and $\delta\mathbf{r}_2 = (\delta x_1 + \delta x_2)\mathbf{i}$

(b) The only applied force is given as $F\mathbf{i}$, which acts on M_2.

(c) From Equation (23) we have

$$\delta W = F\mathbf{i} \cdot \delta\mathbf{r}_2 = F\mathbf{i} \cdot \sum_{j=1}^{2} \frac{\partial \mathbf{r}_2}{\partial q_j} \delta q_j = F\mathbf{i} \cdot (\delta x_1\mathbf{i} + \delta x_2\mathbf{i}) = F\delta x_1 + F\delta x_2 \tag{39}$$

Hence, the generalized forces are: $Q_{x_1} = F$ and $Q_{x_2} = F$. Here it is important to note that the applied force does virtual work on *both* generalized coordinates.
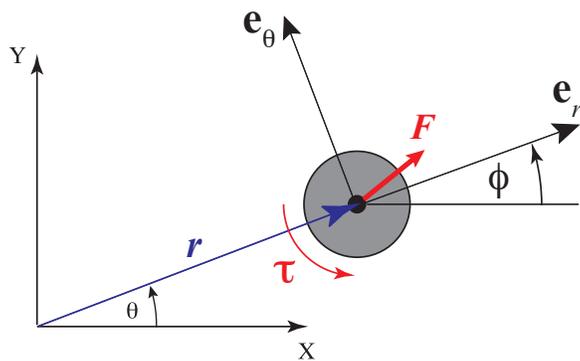
Figure 5. Disk of mass m moving under influence of force $\mathbf{F}$, and torque $\boldsymbol{\tau}$.

Example 2.3 *Consider the system shown in Figure 5. In this example, we have a disk of mass, m, and radius, R, undergoing general motion in the X,Y plane, which is acted upon by a force, $\mathbf{F}$, and torque, $\boldsymbol{\tau}$. Determine the total work done by the generalized forces.*

Solution:

In this example, there is a single body and three degrees of freedom. Thus, we have the same generalized coordinates as in Example 2.1, plus $\phi\mathbf{k}$, and the same force, plus the applied torque, $\tau\mathbf{k}$. Hence, the work done by the applied force is the same as that given by Equation (36), which was obtained using Equation (22), but now we must add the work done by the torque, which is given by

$$\delta W_\phi = \boldsymbol{\tau} \cdot \delta\boldsymbol{\phi} = \tau\delta\phi \tag{40}$$

So, the total work is given by

$$\delta W = \delta W = (4\cos\theta + 3\sin\theta)\delta r + (3\cos\theta - 4\sin\theta)r\delta\theta + \tau\delta\phi \tag{41}$$

In this example, there is only one rotational degree of freedom, so $N_R = 1$, and $\boldsymbol{\theta} = \phi\mathbf{k}$.

3　Lagrange's Equations of Motion

Writing the position and velocity of each particle in the system as a function of the *generalized coordinates* q_i, and their derivatives with respect to time $\dot{q}_i$, we have that $L = L(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n)$. Hence, following the procedure detailed in Section 1, but replacing the functional F with L, and u with q_i, we obtain

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{42}$$

Equation (42) is the Lagrange equation for systems where the virtual work may be expressed as a variation of a potential function, V. In the frequent cases where this is not the case, the so-called *extended Hamilton's principle* must be used.

3.1　Lagrange's Equations Via The Extended Hamilton's Principle

If the virtual work is not derivable from a potential function, then we must begin with equation (18). The kinetic energy is given by

$$T = \frac{1}{2}\sum_{k=1}^{N} m_k \dot{\mathbf{r}}_k \cdot \dot{\mathbf{r}}_k, \tag{43}$$

and the virtual work may be computed in the manner of Equation (15). Care must be taken to account for p forces, and N generalized coordinates. Hence, the variation of the j$^{\text{th}}$ position is given by

$$\delta\mathbf{r}_j = \sum_{k=1}^{N} \frac{\partial \mathbf{r}_j}{\partial q_k}\delta q_k \tag{44}$$

Hence, the virtual work done by p forces acting over N generalized coordinates is given by

$$\delta W = \sum_{j=1}^{p} \mathbf{F}_j \cdot \delta\mathbf{r}_j = \sum_{j=1}^{p} \mathbf{F}_j \cdot \sum_{k=1}^{N} \frac{\partial \mathbf{r}_j}{\partial q_k}\delta q_k. \tag{45}$$

Switching the order of summation, we have

$$\delta W = \sum_{k=1}^{N} \left(\sum_{j=1}^{p} \mathbf{F}_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_k} \right) \delta q_k = \sum_{k=1}^{N} Q_k \delta q_k, \tag{46}$$

where

$$Q_k = \sum_{k=1}^{p} F_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_k}, \text{ for } k = 1, 2, 3, \cdots, N. \tag{47}$$

Substitution of Equation (46) into Equation (18) yields

$$\int_{t_1}^{t_2} \sum_{k=1}^{N} \left[\frac{\partial T}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \right) + Q_k \right] \delta q_k dt = 0. \tag{48}$$

Since the δq_k are arbitrary between t$_1$, and t$_2$, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} = Q_k. \tag{49}$$

If some of the forces are derivable from a potential function V, we may divide the virtual work up into *conservative* virtual work, which is done by those forces derivable from a potential function, and *non--conservative* virtual work done by those which are not derivable from a potential function V. Thus we have,

$$\delta W = \delta W^c + \delta W^{nc} = -\delta V + \sum_{k=1}^{N} Q_k^{nc}\delta q_k. \tag{50}$$

Substitution of Equation (50) into Equation (18) yields

$$\int_{t_1}^{t_2} \delta\,(T-V)\,dt + \int_{t_1}^{t_2} \sum_{k=1}^{N} Q_k^{nc}\delta q_k dt = 0, \tag{51}$$

or, from the definition of the Lagrangian,

$$\int_{t_1}^{t_2} \left(\delta L + \sum_{k=1}^{N} Q_k^{nc}\delta q_k \right) dt. \tag{52}$$

Applying Equation (8), we obtain Lagrange's equation in its most familiar form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_k^{nc}. \tag{53}$$

3.2   Rayleigh's Dissipation function

An important case where a nonconservative force may be derived from a potential function is that of the viscous damping force. The potential function for viscous forces is called the*Rayleigh dissipation function* after Lord Rayleigh. Presented here without derivation, the Rayleigh dissipation function for a single linear viscous damper is given by

$$D = \frac{1}{2}c\dot{x}^2, \tag{54}$$

where c is the damping constant, and x is the displacement from inertial ground. In a system where there are multiple degrees of freedom, and several dampers between the mass particles, the velocity difference between the ends of the dampers must be accounted for. For example, in a two degree of freedom system, with one set of springs and dampers attached to ground, and another set between the two masses, we have

$$D = \frac{1}{2}\left[c_1\dot{x}_1^2 + c_2(\dot{x}_2 - \dot{x}_1)^2 \right], \tag{55}$$

where c_1, and c_2 are the damping constants, and $\dot{x}_1$, and $\dot{x}_2$ are the velocities of the two masses.

In general, Equation (53) may be modified to include the Rayleigh dissipation function, and will assume the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} = Q_k^{nc}. \tag{56}$$

3.3   Kinematic Requirements of Lagrange's Equation

Lagrangian dynamics, as described thus far, provides a very powerful means to determine the equations of motion for complicated discrete (finite degree of freedom) systems. However, there are two primary kinematic requirements which must be achieved before the determination of the potential functions, and subsequent application of Lagrange's equation.

1. *Coordinate choice:*

    (a) The choice of coordinates must be independent and orthogonal. While it is possible to use non-
        -orthogonal coordinates, the additional complexity incurred is not worth the effort in discrete
        models. Examples of orthogonal coordinate choices include: Cartesian – x, y, z, cylindrical – r,
        θ, z, and spherical – r, θ, and ϕ.

    (b) The coordinates must locate the body with respect to an *inertial reference frame.* An inertial
        reference frame is simply one which is not accelerating.

2. *Translational and rotational energy:*

    In rigid bodies, both the translational and rotational kinetic energy must be accounted for. Three
    cases exist:

    (a) *Pure rotation* – An object which is in pure rotation has at least one point or line which has zero
        translational velocity. In this case, all of the kinetic energy is rotational, so only the rotational
        kinetic potential function need be accounted for.

    (b) *Pure translation* – An object is said to be in pure translation if it has no rotation. In this case
        only the translational kinetic potential function need be accounted for, so only the velocity of
        the center of mass is needed.

    (c) *Translation and rotation* – A body which is both translating and rotating exhibits no station-
        ary points as does a body in pure rotation. However, a translating and rotating body can
        exhibit *instantaneous centers of rotation* which have zero velocity with respect to an inertial
        reference for an instant. An important example of this case is rolling without slipping. The
        point of contact between a wheel and the ground has zero velocity, so the kinetic energy may
        be considered to be purely rotational as long as the inertia with respect to the instant center
        is used.

    In the general case of rotation and translation, the *velocity of the center of mass* is used for
    the translational kinetic potential, and the *angular velocity about the mass center* is used to
    determine the rotational kinetic potential. Hence, the inertia about the mass center is used
    in this case. This approach may be used for *all* cases since *all motion* may be broken up into
    rotation about the center of mass, and translation of the center of mass.

    As an example, consider a slender rod of mass, m, and length L used as a pendulum. The
    kinetic energy may be found in one case, using the instant center of rotation approach, to
    be $T = T_{rot} = \frac{1}{2}(\frac{1}{3}mL^2)\dot{\theta}^2$, were the pivot point it the reference for the moment of inertia,
    and in the general case, $T = T_{rot} + T_{trans}$, where the mass center is used. In later case,
    $T_{rot} = \frac{1}{2}(\frac{1}{12}mL^2)\dot{\theta}^2$, and $T_{trans} = \frac{1}{2}mv_{CM}^2 = \frac{1}{2}m(\frac{L}{2}\dot{\theta})^2$. Hence, adding the rotational and
    translational kinetic potentials yields the same result as obtained by using the instant center
    of rotation approach.

3.4   Lagrange Equation Examples

Example 3.1 *Consider the system shown in Figure 6. The equations of motion may be easily found
using Equation 56. In this case $q_1 = x_1$, and $q_2 = x_2$. First, we must find the potential functions.*
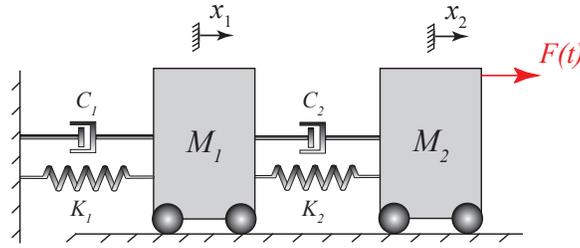
Figure 6. Two degree of freedom example with inertial coordinates.

Solution

Kinetic Energy:

$$T = \frac{1}{2}\left(M_1\dot{x}_1^2 + M_2\dot{x}_2^2\right) \tag{57}$$

Potential Energy:

$$V = \frac{1}{2}\left[K_1 x_1^2 + K_2\left(x_2 - x_1\right)^2\right] \tag{58}$$

Rayleigh's Dissipation Function:

$$D = \frac{1}{2}\left[C_1\dot{x}_1^2 + C_2\left(\dot{x}_2 - \dot{x}_1\right)^2\right] \tag{59}$$

Generalized Force:

$$Q_{x_2} = f(t) \tag{60}$$

Lagrange Function:

$$L = T - V = \frac{1}{2}\left(M_1\dot{x}_1^2 + M_2\dot{x}_2^2\right) - \frac{1}{2}\left[K_1 x_1^2 + K_2\left(x_2 - x_1\right)^2\right] \tag{61}$$

Substitution of the Lagrange function, L, the dissipation function, D, and the generalized force, Q_{x_2} into Equation (56) yields the system equations of motion:

$$M_1\ddot{x}_1 + (C_1 + C_2)\dot{x}_1 - C_2\dot{x}_2 + (K_1 + K_2)x_1 - K_2 x_2 = 0 \tag{62}$$
$$M_2\ddot{x}_2 - C_2\dot{x}_1 + C_2\dot{x}_2 - K_2 x_1 + K_2 x_2 = f(t). \tag{63}$$

Example 3.2 *Consider the system described in Example 2.2; determine the equations of motion using Lagrange's Equation given by Equation (56).*

Solution

Kinematics:

$\mathbf{r}_1 = x_1\mathbf{i}$ and $\mathbf{r}_2 = (x_1 + x_2)\,\mathbf{i}$, thus, $\dot{\mathbf{r}}_1 = \dot{x}_1\mathbf{i}$, and $\dot{\mathbf{r}}_2 = (\dot{x}_1 + \dot{x}_2)\,\mathbf{i}$.

Kinetic Energy:

$$T = \frac{1}{2}M_1\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1 + \frac{1}{2}M_2\dot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2 = \frac{1}{2}\left[M_1\dot{x}_1^2 + M_2\left(\dot{x}_1 + \dot{x}_2\right)^2\right] \tag{64}$$

Potential Energy:

$$V = \frac{1}{2}\left[K_1 x_1^2 + K_2 x_2^2 + K_3\left(x_1 + x_2\right)^2\right], \tag{65}$$

where it should be noted that the displacement of the left end of the third spring is due to the inertial displacement of M_2 because its right end is grounded, whereas the displacement of the second spring is due only to the relative separation of the two masses as described by x_2 (rel. to x_1).

Rayleigh's Dissipation Function:

The form of the dissipation function follows that of the potential energy:

$$D = \frac{1}{2}\left[C_1 \dot{x}_1^2 + C_2 \dot{x}_2^2 + C_3\left(\dot{x}_1 + \dot{x}_2\right)^2\right]. \tag{66}$$

Generalized Forces:

The generalized forces have already been identified in Example 2.2 as

$$Q_{x_1} = F \text{ and } Q_{x_2} = F \tag{67}$$

Lagrange Function:

$$L = T - V = L = \tfrac{1}{2}\left[M_1 \dot{x}_1^2 + M_2\left(\dot{x}_1 + \dot{x}_2\right)^2\right] - \tfrac{1}{2}\left[K_1 x_1^2 + K_2 x_2^2 + K_3\left(x_1 + x_2\right)^2\right] \tag{68}$$

Application of Lagrange's equation yields:

$$\left(M_1 + M_2\right)\ddot{x}_1 + M_2 \ddot{x}_2 + (C_1 + C_3)\dot{x}_1 + C_3 \dot{x}_2 + (K_1 + K_3)x_1 + K_3 x_2 = F, \tag{69}$$

and

$$M_1 \ddot{x}_2 + M_2 \ddot{x}_2 + C_3 \dot{x}_1 + \left(C_2 + C_3\right)\dot{x}_2 + K_3 x_1 + \left(K_2 + K_3\right)x_2 = F. \tag{70}$$

The use of inertial coordinates in this example would considerably simplify the resulting EOMs. Substitution of the inertial coordinates $y_1 = x_1$ and $y_2 = y_1 + x_2 \implies x_2 = y_2 - y_2$, and some algebra yields:

$$M_1 \ddot{y}_1 + (C_1 + C_2)\dot{y}_1 - C_2 \dot{y}_2 + (K_1 + K_2)y_1 - K_2 y_2 = 0 \tag{71}$$

and

$$M_2 \ddot{y}_2 - C_2 \dot{y}_1 + (C_2 + C_3)\dot{y}_2 - K_2 y_1 + (K_2 + K_3)y_2 = F \tag{72}$$

This form of the EOMs is said to be *viscoelastically coupled* and *inertially uncoupled*. Recasting the resulting EOMs into matrix form, we have:

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} + \begin{bmatrix} C_1 + C_2 & -C_2 \\ -C_2 & C_2 + C_3 \end{bmatrix}\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 + K_3 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix} \tag{73}$$

4   Constrained Maxima and Minima

There are two principal advantages in using the analytical approach to determining the equation of motion for a dynamical system:

1. only positions and velocities need be determined. The resulting accelerations are determined automatically. This often means a considerable savings in computation.

2. All work-less constraint forces are automatically eliminated from the calculations. In the Newtonian approach, all contact forces applied to the body in question must be accounted for, and ultimately determined in order to solve the resulting equations of motion. In a kinematic chain, this requirement leads to significant effort.

The question arises, however: what if we need to know the constraint forces in a system? This is often the case in engineering design and analysis. Fortunately, Lagrange developed an elegant means to solve constrained problems of extremum in general which yields only the constraint forces of interest in dynamical systems.

4.1   Lagrange's Multipliers

Consider a function $u = f(x_1, x_2, \ldots, x_n)$, having at least two continuous derivatives with respect to the independent variables, to be extremized. Obviously, the function, u, must be at least of quadratic polynomial order, or it cannot exhibit an extremum! A necessary condition for extremum is that the total differential of the function u vanishes [3];

$$du = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots, + \frac{\partial f}{\partial x_n}dx_n = 0. \tag{74}$$

Assuming that the variables, x_i, are independent, it follows that the sufficient condition for extremum is

$$\frac{\partial f}{\partial x_1} = 0,$$
$$\frac{\partial f}{\partial x_2} = 0,$$
$$\vdots$$
$$\frac{\partial f}{\partial x_n} = 0. \tag{75}$$

Next, consider the case where some of the independent variables are related by constraints. It is easy to show that Equation (74) is still valid. Consider the function, $u = f(x, y, z)$, where $z = z(x, y)$, is related to x, and y through a constraint equation of the form

$$\phi(x, y, z) = 0. \tag{76}$$

Considering the variables x, and y, as the independent variables, the necessary conditions for extremum are

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0,$$
$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0. \tag{77}$$

Hence, the total differential becomes

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right) = 0. \tag{78}$$

Since

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy, \tag{79}$$

it follows that

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0. \tag{80}$$

Hence, Equation (74) is still valid even when there are constraint relations between the independent variables. Next, consider the total differential of the constraint given by Equation (76),

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0. \tag{81}$$

Multiplying Equation (81) by an undetermined multiplier, λ, and adding it to Equation (80) yields

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0. \tag{82}$$

The multiplier, λ, may be chosen so that

$$\begin{aligned}\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} &= 0 \\ \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} &= 0 \\ \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} &= 0 \\ \phi(x,y,z) &= 0,\end{aligned} \tag{83}$$

so that the necessary condition for an extremum of $u = f(x, y, z)$ is satisfied.
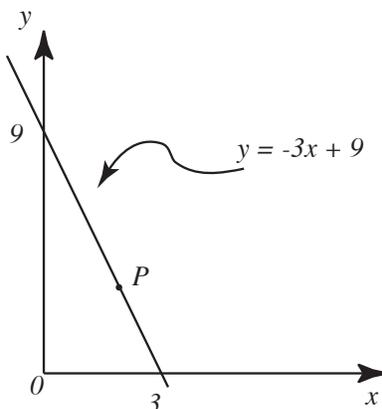
Figure 7. Analytical geometry example.

As an example of the application of Lagrangian multipliers, consider the problem of finding the coordinates of the nearest point to the origin, P, on a specified line [3]. The function to be extremized is the squared distance to the point given by

$$f(x, y) = r^2 = x^2 + y^2, \tag{84}$$

subject to the constraint

$$\phi(x, y) = y + 3x - 9 = 0. \tag{85}$$

We note here that $f(x, y)$ is of class C^2, whereas $x + y$, which is also a measure of the distance from the origin to the point P, is not. Alternatively, we could also use the Euclidian norm, $r = \sqrt{x^2 + y^2}$, as the

15

distance measure to be minimized, but the computations are slightly more complex. Applying Equations (83), we have

$$2x + 3\lambda = 0 \tag{86}$$
$$2y + \lambda = 0 \tag{87}$$
$$y + 3x - 9 = 0. \tag{88}$$

From Equation (87), we have $\lambda = -2y$. Substitution into (86) yields $y = \frac{1}{3}x$. Substitution of this result into (88) yields $x = \frac{27}{10}$, and $y = \frac{9}{10}$. Hence, the point $P = (\frac{27}{10}, \frac{9}{10})$ is the nearest point to the origin on the line given by $y = -3x + 9$. That this is so may be easily verified by taking the dot product of the vector from the origin to P with that in the same direction as the line to demonstrate that they are indeed perpendicular:

$$(\frac{27}{10}, \frac{9}{10}) \cdot (3, -9) = 0. \tag{89}$$

A mathematical shorthand may be employed to include the constraints and the function to be extremized in a single "augmented" function given by

$$f^* = f(x_1, x_2, \ldots, x_n) + \lambda\phi(x_1, x_2, \ldots, x_n), \tag{90}$$

where $\phi(x_1, x_2, \ldots, x_n)$ is the constraint function. Next, taking the total differential of Equation (90) while considering the Lagrange multiplier constant, and by setting the coefficients of the differentials to zero, we arrive at

$$\frac{\partial f}{\partial x_1} + \lambda\frac{\partial \phi}{\partial x_1} = 0$$
$$\frac{\partial f}{\partial x_2} + \lambda\frac{\partial \phi}{\partial x_2} = 0$$
$$\vdots$$
$$\frac{\partial f}{\partial x_n} + \lambda\frac{\partial \phi}{\partial x_n} = 0$$
$$\phi(x_1, x_2, \ldots, x_n) = 0, \tag{91}$$

In the case where there are n variables related by m constraints, we must define m Lagrange multipliers. Hence, Equation (90) becomes

$$f^* = f(x_1, x_2, \ldots, x_n) + \sum_{j=1}^{m} \lambda_j\phi_j(x_1, x_2, \ldots, x_n), \tag{92}$$

and we define $n + m$ equations of the form

$$\frac{\partial f}{\partial x_i} + \sum_{j=1}^{m} \lambda_j\frac{\partial \phi_j}{\partial x_i} = 0, \text{ for } i = 1, 2, \ldots, n, \tag{93}$$

and

$$\phi_j(x_1, x_2, \ldots, x_n) = 0, \text{ for } j = 1, 2, \ldots, m. \tag{94}$$

4.2   Application of Lagrange Multipliers to Compute Equilibrium Reaction Forces

Next, we consider the application of Lagrange multipliers to determine the static reaction forces at equilibrium.

Theorem 4.1 *The total work done by the forces acting on a body in equilibrium during a reversible virtual displacement consistent with the system constraints is zero.*

The above theorem, stated here without proof, seems reasonable since equilibrium implies the absence of explicit time dependent forces. Consider the case for a conservative system where

$$\delta W = -\delta V \tag{95}$$

At equilibrium, we have that

$$\delta W = -\delta V = 0. \tag{96}$$

Consider the potential energy to be a function of n coordinates, $x_1, x_2, \ldots, x_n$, such that

$$\delta V = \frac{\partial V}{\partial x_1} \delta x_1 + \frac{\partial V}{\partial x_2} \delta x_2 + \ldots + \frac{\partial V}{\partial x_n} \delta x_n = 0. \tag{97}$$

Next, we consider the same system to be subject to a constraint of the form

$$\phi(x_1, x_2, \ldots, x_n) = 0. \tag{98}$$

Taking the total variation of the constraint, we have

$$\delta \phi = \frac{\partial \phi}{\partial x_1} \delta x_1 + \frac{\partial \phi}{\partial x_2} \delta x_2 + \ldots + \frac{\partial \phi}{\partial x_n} \delta x_n = 0. \tag{99}$$

Multiplying Equation (99) by an unknown Lagrange multiplier, and *subtracting*[4] it from Equation (97) we have

$$\left(\frac{\partial V}{\partial x_1} - \lambda \frac{\partial \phi}{\partial x_1} \right) \delta x_1 + \left(\frac{\partial V}{\partial x_2} - \lambda \frac{\partial \phi}{\partial x_2} \right) \delta x_2 + \ldots + \left(\frac{\partial V}{\partial x_n} - \lambda \frac{\partial \phi}{\partial x_n} \right) \delta x_n = 0. \tag{100}$$

Equation (100) is analogous to Equation (82), and again, a system of $n + 1$ equations analogous to Equation (83) results. As before, mathematical shorthand may be used to augment the potential energy, so that it assumes the form

$$V^* = V - \lambda \phi. \tag{101}$$

The variation is then taken as usual, but the Lagrange multiplier, λ, is assumed to be constant. Consider the following simple example:

Given the simple pendulum shown in Figure 8, determine the reaction force on the pivot, F_p, as well as the conditions of equilibrium using the Lagrange multiplier method.

Solution: Taking the pivot point as datum, we find that the potential energy is given by

$$V = -mgr \cos \theta,$$

subject to the constraint

$$\phi = r - l = 0.$$

The augmented potential energy is given by

---

[4]The variation of the constraint is subtracted, since it represents virtual work done by constraint forces *which always oppose motion*. Where Newton's second law is applied, such forces would carry a negative sign on the side of the equation of motion opposite that of the inertial reaction – i.e. the $m\mathbf{a}$ or $\mathbf{I}\alpha$ term.
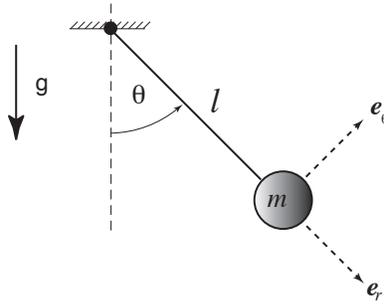
Figure 8. Simple pendulum example.

$$V^* = V - \lambda\phi = -mgr\cos\theta - \lambda(r - l)$$

Taking the variation of V^*, we have

$$\delta V^* = -mg\cos\bar{\theta}\delta r + mgr\sin\bar{\theta}\delta\theta - \lambda\delta r = 0$$

Since the variations are independent, we must have

$$-\lambda - mg\cos\bar{\theta} = 0$$
$$\sin\bar{\theta} = 0,$$

where $\bar{\theta}$ is the value of θ at equilibrium. Thus, we have: $\bar{\theta} = 0$ and $\bar{\theta} = \pi$, so $\lambda = -mg\cos\bar{\theta} = \mp mg$ at the equilibrium angles, $\bar{\theta} = 0$ and $\bar{\theta} = \pi$, respectively. In this case, λ is the force of constraint acting on the pendulum mass in the r-direction because the constraint was in the r-direction, so it is positive when $\bar{\theta} = \pi$ because it acts in the $+r$-direction to overcome the weight of the particle, and negative when $\bar{\theta} = 0$, because supporting the weight requires the constraint force to act in the $-r$-direction.

4.3 Application of Lagrange Multipliers to Compute Dynamic Reaction Forces

The Lagrange multiplier method readily extends to the non-equilibrium dynamic case. for simple geometric constraints such as illustrated in the previous section, Equation (101), still applies, and is used to form the so-called *augmented Lagrangian*:

$$L^* = T - V^*. \tag{102}$$

The equations of motion are then determined as usual.

4.3.1 Gear Train Constraint Force Example

Consider the system shown in Figure 9 which represents a simple gear train connecting two inertias – assumed to account for the gear inertias as well. The gears have radii, r_1 and r_2, and posses N_1, and N_2 teeth, respectively. It is desired to know the constraint force between the gears due to the application of an applied torque, $\tau(t)$, acting on the J_1 inertia. For the gears to mesh, the geometric constraint may be stated as

$$N_2\theta_2 = N_1\theta_1 \tag{103}$$

Hence, if we want the resulting equation of motion in terms of θ_1, the constraint function could be written

$$\phi(\theta_1, \theta_2) = \theta_2 - \frac{N_1}{N_2}\theta_1. \tag{104}$$
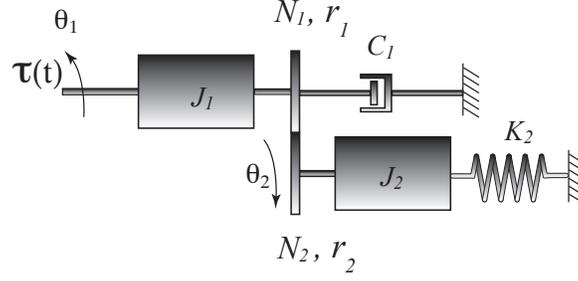
Figure 9. Gear train with viscoelastic constraints.

The kinetic and augmented potential energies may then be expressed as:

$$T = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2, \tag{105}$$

and

$$V^* = \frac{1}{2}K_2\theta_2^2 - \lambda\left(\theta_2 - \frac{N_1}{N_2}\theta_1\right). \tag{106}$$

The Rayleigh's dissipation function is given by:

$$D = \frac{1}{2}C_1\dot{\theta}_1. \tag{107}$$

Hence, the augmented Lagrangian becomes

$$L^* = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2 - \frac{1}{2}K_2\theta_2^2 + \lambda\left(\theta_2 - \frac{N_1}{N_2}\theta_1\right). \tag{108}$$

Application of Lagrange's equation on the θ_1 coordinate yields:

$$J_1\ddot{\theta}_1 + C_1\dot{\theta}_1 - \frac{N_1}{N_2}\lambda = \tau(t) \tag{109}$$

Likewise, for θ_2, we obtain

$$J_2\ddot{\theta}_2 + K_2\theta_2 - \lambda = 0 \tag{110}$$

Substitution of the constraint given by Equation (103) into Equation (110) yields

$$\frac{N_1}{N_2}\left(J_2\ddot{\theta}_1 + K_2\theta_1\right) - \lambda = 0. \tag{111}$$

Thus, the constraint torque acting on the gear connected to inertia, J_2, required to force the gears to mesh is given by

$$\lambda = \frac{N_1}{N_2}\left(J_2\ddot{\theta}_1 + K_2\theta_1\right). \tag{112}$$

Substitution of Equation (112) into (109) yields

$$\left(J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2\right)\ddot{\theta}_1 + C_1\dot{\theta}_1 + \left(\frac{N_1}{N_2}\right)^2 K_2\theta_1 = \tau(t). \tag{113}$$

Equation (113) may be solved to obtain $\theta_1(t)$, and the result substituted into Equation (112) to obtain the constraint torque, λ. To determine the constraint force between the gears, we note that the torque, λ, is acting on the J_2 inertia, so the constraint force is given by

$$f_c = \frac{\lambda}{r_2} = \frac{N_1}{r_2 N_2}\left(J_2\ddot{\theta}_1 + K_2\theta_1\right). \tag{114}$$

From Newton's third law, the force acting on the gear connected to J_1 would have the same magnitude, but opposite sign.

The approach illustrated in this example provides a systematic method to analyze much more complicated gear trains. For example, Mantriota et al. utilize this approach to predict the efficiency of epicyclic gear trains [4].

5   Hamilton's Principle: Continuous Models

Hamilton's principle may be used to obtain the equations of motion for continuous system models in much the same manner as applied to discrete systems. For example, the elastic potential energy in a tight string with tension, τ Newtons, and mass per unit length, ρ kg/m, is given by

$$V = \frac{1}{2}\tau \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \tag{115}$$

Likewise, the kinetic energy is given by

$$T = \frac{1}{2}\rho \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx \tag{116}$$

The extended Hamilton's principle may be written as

$$\delta \int_{t_1}^{t_2} [T - V]\, dt = \int_{t_1}^{t_2} \delta W\, dt \tag{117}$$

where δW denotes the virtual work done by string against non-conservative forces, and δ is the variational operator or derivative (assumed to occur contemporaneously). For our string example, we thus have

$$\delta \int_{t_1}^{t_2} \int_0^L \left[\frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau\left(\frac{\partial u}{\partial x}\right)^2\right] dx\, dt = \int_{t_1}^{t_2} \int_0^L \left[-f(x,t)\delta u + \gamma\frac{\partial u}{\partial t}\delta u\right] dx\, dt \tag{118}$$

where the negative sign on $f(x,t)$ denotes that it is adding energy by forcing the string in the direction of increasing displacement instead of absorbing energy like the damping component (with coefficient, γ N-s/ m^2) does. This is merely a restatement of the first law of thermodynamics. Taking the variation under the integral sign, we have

$$\int_{t_1}^{t_2} \int_0^L \left[\rho\frac{\partial u}{\partial t}\delta\frac{\partial u}{\partial t} - \tau\frac{\partial u}{\partial x}\delta\frac{\partial u}{\partial x}\right] dx = \int_{t_1}^{t_2} \int_0^L \left[-f(x,t)\delta u + \gamma\frac{\partial u}{\partial t}\delta u\right] dx\, dt \tag{119}$$

Since the order of differentiation does not matter, we can switch the order of the variation and the partial derivatives to obtain

$$\int_{t_1}^{t_2} \int_0^L \left[\rho\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} - \tau\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\right] dx\, dt = \int_{t_1}^{t_2} \int_0^L \left[-f(x,t)\delta u + \gamma\frac{\partial u}{\partial t}\delta u\right] dx\, dt \tag{120}$$

Integrating the first term by parts with respect to t yields

$$\int_{t_1}^{t_2} \int_0^L \rho\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} dx\, dt = \int_0^L \left[\rho\frac{\partial u}{\partial t}\delta u\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \rho\frac{\partial^2 u}{\partial t^2}\delta u\right] dx\, dt \tag{121}$$

The first term on the rhs of Equation (121) vanishes due to the stipulation that variation must vanish at the arbitrary times, t_1, and t_2 – i.e. $\delta u(x,t_1) = \delta u(x,t_2) = 0$ See also: Section 1.1. Hence, Equation (120) becomes

$$\int_{t_1}^{t_2} \int_0^L \left[-\rho\frac{\partial^2 u}{\partial t^2}\delta u - \tau\frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\right] dx\, dt = \int_{t_1}^{t_2} \int_0^L \left[-f(x,t)\delta u + \gamma\frac{\partial u}{\partial t}\delta u\right] dx\, dt \tag{122}$$

Similarly, integrating the second term in Equation (120) by parts with respect to x, we have

$$-\int_{t_1}^{t_2}\int_0^L \tau \frac{\partial u}{\partial x}\frac{\partial \delta u}{\partial x}\,dx\,dt = \int_{t_1}^{t_2}\left[\tau\int_0^L \frac{\partial^2 u}{\partial x^2}\delta u\,dx - \tau\frac{\partial u}{\partial x}\delta u|_0^L\right]dt \tag{123}$$

Substitution of (123) into (122) yields

$$\int_{t_1}^{t_2}\left\{\int_0^L\left[\tau\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} + f(x,t) - \gamma\frac{\partial u}{\partial t}\right]\delta u\,dx + \tau\frac{\partial u(0,t)}{\partial x}\delta u(0,t) - \tau\frac{\partial u(L,t)}{\partial x}\delta u(L,t)\right\}dt = 0 \tag{124}$$

The variations must be consistent with the boundary conditions – i.e. if the boundary condition vanishes, then the variation of the boundary condition also vanishes – so the admissible boundary conditions for general homogeneous boundary conditions are:

1. *fixed boundaries*: $u(0,t) = u(L,t) = 0$ or

2. *roller-constrained boundaries*: $\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x} = 0$. In general, mixed combinations of the so-called *Dirichlet* (displacement) and *Neumann* (strain) boundary conditions may apply as well.[5]

In our fixed-fixed string example, the first set of displacement boundary conditions apply. Hence, applying the applicable boundary conditions, Equation (124) becomes

$$\int_{t_1}^{t_2}\int_0^L\left[\tau\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} + f(x,t) - \gamma\frac{\partial u}{\partial t}\right]\delta u\,dx\,dt = 0 \tag{125}$$

Thus, by the fundamental lemma of the calculus of variations, since the variation δu is arbitrary, except on the boundaries and at the arbitrary times, t_1, and t_2, we must have

$$-\tau\frac{\partial^2 u}{\partial x^2} + \rho\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial u}{\partial t} = f(x,t) \tag{126}$$

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[5]See: http://en.wikipedia.org/wiki/Dirichlet_boundary_condition and http://en.wikipedia.org/wiki/Neumann_problem
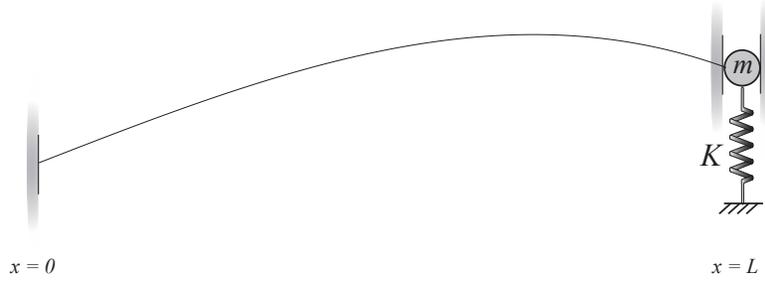
Figure 10. String with left end fixed and right end bearing an elastically-supported roller of mass, m.

Example 5.1 *Consider the string shown in Figure 10, and assuming the presence of distributed viscous damping characterized by equivalent viscous damping coefficient, γ, as described in Section 5, apply Hamilton's principle to determine the equation of motion and boundary conditions.*

Solution:

Equations (115) and (116) are still valid, but the additional kinetic and potential energy contributed by the boundary mass and spring must be included. Hence, we have:

$$V = \frac{1}{2}\tau \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{1}{2}Ku^2(L,t) \tag{127}$$

Likewise, the kinetic energy is given by

$$T = \frac{1}{2}\rho \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}M\dot{u}^2(L,t) \tag{128}$$

where the $\dot{u} = \partial u/\partial t$ notation has been used on the boundary kinetic energy term for convenience of notation while explicitly indicating that pertains to the boundary point, $x = L$.

The nonconservative virtual work done by the system against the equivalent viscous damping is given by

$$\delta W_{nc} = \gamma \int_0^L \frac{\partial u}{\partial t}\delta u \, dx \tag{129}$$

Hence,

$$\delta \int_{t_1}^{t_2} [T - V] \, dt = \delta \int_{t_1}^{t_2} \left\{ \int_0^L \left[\frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau\left(\frac{\partial u}{\partial x}\right)^2\right] dx + \frac{1}{2}M\dot{u}^2(L,t) - \frac{1}{2}Ku^2(L,t) \right\} dt =$$

$$\gamma \int_0^L \frac{\partial u}{\partial t}\delta u \, dx \tag{130}$$

Taking the variation under the integrals and integrating by parts as before yields:

$$\int_{t_1}^{t_2} \left\{ \int_0^L \left[\tau\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} - \gamma\frac{\partial u}{\partial t}\right] \delta u \, dx + \tau\frac{\partial u(0,t)}{\partial x}\delta u(0,t) \right.$$

$$\left. - \left(M\ddot{u}(L,t) + Ku(L,t) + \tau\frac{\partial u(L,t)}{\partial x}\right)\delta u(L,t) \right\} dt = 0 \tag{131}$$

22

Since all of the variations are arbitrary and independent, except for the variation of the fixed left boundary which must vanish to be consistent with zero displacement, we obtain:

$$-\tau\frac{\partial^2 u}{\partial x^2} + \rho\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial u}{\partial t} = 0 \tag{132}$$

$$u(0,t) = 0 \tag{133}$$

and

$$M\ddot{u}(L,t) + Ku(L,t) + \tau\frac{\partial u(L,t)}{\partial x} = 0 \tag{134}$$

6 Bibliography

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