A Brief Introduction To Laplace Transformation

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1 Linear System Modeling Using Laplace Transformation

Laplace transformation provides a powerful means to solve linear ordinary differential equations in the time domain, by converting these differential equations into algebraic equations. These may then be solved and the results inverse transformed back into the time domain. Tables of Laplace transforms are available to facilitate this operation.

Laplace transformation belongs to a general area of mathematics called *operational calculus* which focuses on the analysis of linear systems.

1.1 Laplace Transformation

Laplace transformation belongs to a class of analysis methods called integral transformation which are studied in the field of operational calculus. These methods include the Fourier transform, the Mellin transform, etc. In each method, the idea is to transform a difficult problem into an easy problem. For example, taking the Laplace transform of both sides of a linear, ODE results in an algebraic problem. Solving algebraic equations is usually easier than solving differential equations. The one-sided Laplace transform which we are used to is defined by equation (1), and is valid over the interval $[0, \infty)$. This means that the domain of integration includes its left end point. This is why most authors use the term 0^- to represent the bottom limit of the Laplace integral.

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt \tag{1}$$

The key thing to note is that Equation (1) is not a function of time, but rather a function of the Laplace variable $s = \sigma + j\omega$. Also, the Laplace transform only transforms functions defined over the interval $[0, \infty)$, so *any part of the function which exists at negative values of t is lost!* One of the most useful Laplace transformation theorems is the differentiation theorem.

Theorem 1 *The Laplace transform of the first derivative of a function f is given by*

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-) \tag{2}$$

Proof 1 *Integration of Equation (2) by parts yields*

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{df(t)}{dt}e^{-st}dt$$

$$= f(0^-)e^{-st}\big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} f(t)e^{-st}dt$$

$$= sF(s) - f(0^-)\,\| \tag{3}$$

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By repeating the integration by parts, higher derivatives may be similarly transformed. Thus given

$$A\ddot{x} + B\dot{x} + Cx = f(t) \tag{4}$$

we have by taking the Laplace Transform of both sides of Equation (4)

$$A\left(s^2X(s) - sx\left(0^-\right) - \dot{x}\left(0^-\right)\right) + B\left(sX(s) - x\left(0^-\right)\right) + CX(s) = F(s) \tag{5}$$

or

$$As^2X(s) + BsX(s) + CX(s) = F(s) + A\left(sx\left(0^-\right) + \dot{x}\left(0^-\right)\right) + Bx\left(0^-\right) \tag{6}$$

The most commonly used form of Laplace transformation requires that the forcing function, f(t), be *causal*. Such functions are said to "turn on" just *after $t = 0$*. A causal forcing function has the effect of rendering the forced or particular solution of the ODE independent of the initial conditions, which are considered to exist immediately prior to the application of the applied forcing function. A causal function may be represented by a suffixing a Heaviside step function (denoted in this document as $H(t)$) to it[1]. The Heaviside step function is very convenient to use to represent discontinuous forcing. The Dirac delta is another important function (or distribution) which is often used to represent impulsive forcing.

Definition 1

$$H(t) = \begin{cases} 1 \ for \ t > 0 \\ 0 \ for \ t \leq 0 \end{cases} \tag{7}$$

Definition 2

$$\delta(t) = \begin{cases} \infty \ for \ t = 0^+ \\ 0 \ otherwise \end{cases} \tag{8}$$

Hence, the Heaviside step function "turns on" at the right edge $(t = 0^+)$ of zero, and the Dirac delta function turns on and off at the same place. An additional property of the Dirac delta function is

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{9}$$

Hence, the area under the "curve" defined by the Dirac delta, or impulse function is unity. The unit-step and the Dirac delta function are derivative and anti-derivative of one another.

$$\delta(t) = \frac{dH(t)}{dt} \tag{10}$$

Both the unit-step and Dirac delta belong to a class of functions called generalized functions. The term "generalized" stems from the fact that these functions don't satisfy the continuity requirements

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[1]The Heaviside step function is named for the great applied mathematician/phsysist, Oliver Heaviside. See: http://en.wikipedia.org/wiki/Oliver_Heaviside

of regular functions. This fact not withstanding, however, we may define the following relationships:

$$\frac{d\delta(t)}{dt} = \frac{d^2 H(t)}{dt^2}$$

$$\frac{d^2 \delta(t)}{dt^2} = \frac{d^3 H(t)}{dt^3}$$

$$\vdots$$

$$\frac{d^n \delta(t)}{dt^n} = \frac{d^{n+1} H(t)}{dt^{n+1}} \tag{11}$$

The Laplace transform of the Dirac delta function is revealed by taking the Laplace transform of both sides of equation (10):

$$\mathcal{L}\left\{\delta(t)\right\} = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = \mathcal{L}\left\{\frac{dH(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dH(t)}{dt}e^{-st}dt$$

$$= H(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} H(t)e^{-st}dt$$

$$= 0 + s\left(\int_{0^-}^{0^+}(0)\,e^{-st} + \int_{0^+}^{\infty}(1)\,e^{-st}\right) = 1 \tag{12}$$

1.2   Additional properties of the Dirac delta function

Filtering Property:

$$\int_a^c f(t)\delta(t-b)dt = f(b) \tag{13}$$

for $a < b < c$.

The filtering property may be proved as follows:

Proof 2 (Filtering Property of the Dirac Delta) *Equation (13) may be proved for $a < b < c$ by noting that*

$$\int_a^c \delta(t-b)f(t)dt = \lim_{\varepsilon \to 0}\left[\int_a^{b-\varepsilon}(0)f(t)dt + \int_{b-\varepsilon}^{b+\varepsilon}\delta(t-b)f(t)dt + \int_{b+\varepsilon}^c (0)f(t)dt\right]$$

$$= \lim_{\varepsilon \to 0}\int_{b-\varepsilon}^{b+\varepsilon}\delta(t-b)f(t)dt \tag{14}$$

Letting $\tau = t - b$, integrating equation (14) by parts, and noting equation (10), we have

$$\lim_{\varepsilon \to 0}\int_{b-\varepsilon}^{b+\varepsilon}\delta(t-b)f(t)\,dt = \lim_{\varepsilon \to 0}\int_{-\varepsilon}^{+\varepsilon}\delta(\tau)f(\tau+b)\,d\tau$$

$$= \lim_{\varepsilon \to 0}\left[H(\tau)f(\tau+b)|_{-\varepsilon}^{+\varepsilon} - \int_{-\varepsilon}^{+\varepsilon}H(\tau)f'(\tau+b)\,d\tau\right] \tag{15}$$

The last term in the brackets in Equation (15) vanishes because $H(t)$ is zero in the interval of integration. Thus, we have

$$\lim_{\varepsilon \to 0}\int_{b-\varepsilon}^{b+\varepsilon}\delta(t-b)f(t)\,dt = \lim_{\varepsilon \to 0}\left[H(+\varepsilon)f(b+\varepsilon) - H(-\varepsilon)f(b-\varepsilon)\right] \tag{16}$$

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However, since the limit of $H(t)$ as t approaches zero from the left is zero, and the limit of $H(t)$ as t approaches zero from the right is 1, we have

$$\lim_{\varepsilon \to 0} \int_{b-\varepsilon}^{b+\varepsilon} \delta(t-b)f(t)\,dt = \lim_{\varepsilon \to 0} H(+\varepsilon)f(b+\varepsilon) = H(0^+)f(b^+) = f(b) \tag{17}$$

as claimed. ∎

The filtering property turns out to be extremely useful in both analytical and experimental vibration analysis, as well as many other areas in applied mathematics. An additional useful theorem immediately follows from the filtering property; for all finite values of a:

$$\int_{-\infty}^{\infty} \delta'(x-a)f(x)dx = -f'(a) \tag{18}$$

where the prime indicates differentiation with respect to x. In general, for the nth derivative of the Dirac delta, we have

$$\int_{-\infty}^{\infty} \delta^{(n)}(x-a)f(x)dx = (-1)^n f^{(n)}(a) \tag{19}$$

which may be easily proven via integration by parts.

1.3 Use of Dirac Delta Function in Differential Equations

In the context of an equation of motion where forces or moments are summed to equal and inertial force or moment ($\Sigma \mathbf{F} = m\mathbf{a}$), the Dirac delta function may be thought to carry the units of 1/time, or 1/area, etc. In fact, it may be, and is, thought of as a distribution – albeit a very narrow one. In other words, the force imparted by the delta function is proportional to its integrated area. This means that the Dirac delta function is only useful to describe point-wise or impulsive forces when integrated over the applicable independent variable. Thus, integral transform techniques, such as the Laplace transform, provide the most natural means to utilize the Dirac delta function. The shifting and filtering properties are useful in specifying the effect of an impulsive force applied to a body which may already be in motion.
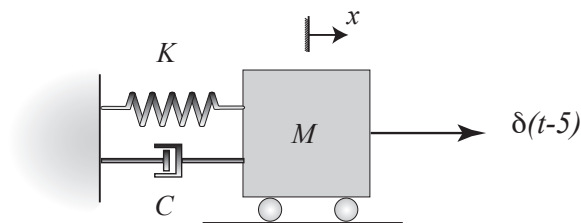

Figure 1. Impulsively forced spring-mass-damper system – use Laplace transformation.

Example 1 *Consider the system shown in Figure 1, which consists of a 1 kg mass restrained by a linear spring of stiffness $K = 10$ N/m, and a damper with damping constant $B = 2$ N-s/m. The system is forced at time $t = 5$ seconds by an impulsive force of magnitude 10 N-s. The system has initial conditions $x(0) = 1$ m, and $\dot{x}(0) = 0$.*
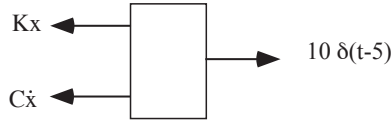
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Figure 2. System free-body diagram.

Solution:

The free body diagram of the system is shown in Figure 2. Writing the equation of motion, we obtain

$$\ddot{x} + 2\dot{x} + 10x = 10\delta(t-5) \tag{20}$$

Taking the Laplace transform of Equation (20), we have

$$\left(s^2 + 2s + 10\right) X(s) = 10e^{-5s} + s + 2 \tag{21}$$

so

$$X(s) = \frac{10e^{-5s} + s + 2}{s^2 + 2s + 10} \tag{22}$$

Taking the inverse Laplace transform of Equation (22), we obtain [you should show this]

$$x(t) = \frac{10}{3}e^{-(t-5)}\sin\left(3\left(t-5\right)\right) H(t-5) + e^{-t}\left(\cos\left(3t\right) + \frac{1}{3}\sin\left(3t\right)\right) \tag{23}$$
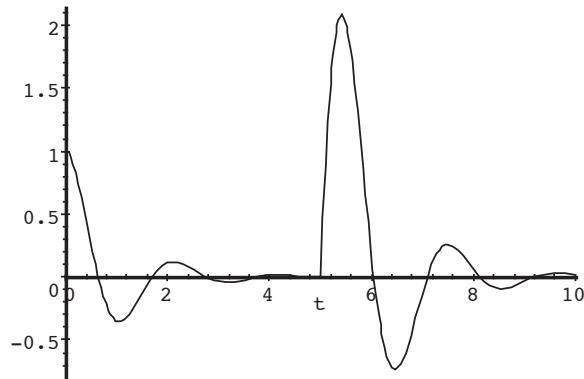

Figure 3. System response to time-shifted impulse

The time response of the system is shown in Figure 3. As you can see, the displacement of the mass has decayed nearly to zero when the impulsive force is applied – causing additional motion.

2 Additional Laplace Theorems

The following is a list of the other more commonly used Laplace transform theorems.

Theorem 2 *Time differentiation:*

$$\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{k-1}\left(0^-\right) \tag{24}$$

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Theorem 3 *Time integration:*

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s) \tag{25}$$

Theorem 4 *Frequency differentiation:*

$$\mathcal{L}\left\{t^n F(t)\right\} = (-1)^n \frac{d^n F(s)}{ds^n} \tag{26}$$

Theorem 5 *Frequency shift:*

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a) \tag{27}$$

Theorem 6 *Time shift:*

$$\mathcal{L}\left\{f(t-T)H(t-T)\right\} = e^{-sT}F(s) \tag{28}$$

3 Dealing With Nonlinearities

The concept of linearity is one of the most important ideas in engineering, physics, and applied mathematics. The difference between a linear and nonlinear system is often the difference between finding a solution or not. Recently, in the past 60 years or so, engineers have become aware of the existence of chaos in nonlinear systems. This awareness has spawned a whole new branch of applied mathematics – the study of chaotic systems. Unfortunately, about 99% of the available mathematical techniques developed in the last 300 years are only useful for linear systems! What are we going to do about this limitation? Well, we're going to linearize the nonlinear terms in our mathematical models as we encounter them. But first, let us carefully define the term linear.

Definition 3 *A function, or operation $f(x)$ is said to be linear if and only if for $f(x_1) = y_1$, and for $f(x_2) = y_2$, we have $f(x_1 + x_2) = y_1 + y_2$. In addition, we must have that $f(kx) = kf(x)$ for any constant k. Definition 2.3 is somewhat more restrictive than the definition of a line [Why?] What do we mean by the term linear operation? The most common linear operations that you will encounter in this course are integration, including Laplace transformation, and differentiation. What are the consequences of linearizing a nonlinear function or operation?*
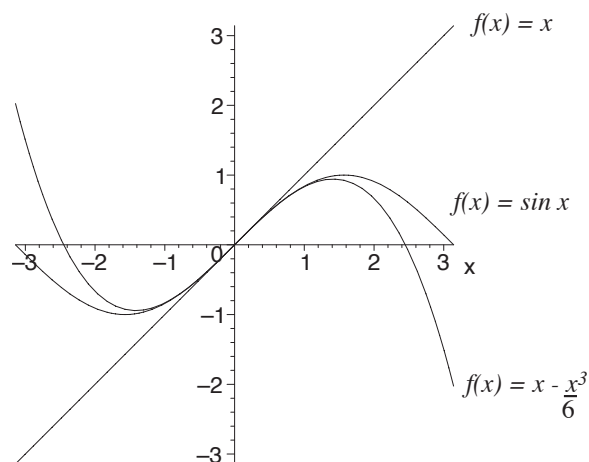

Figure 4. Effect of linearization.

Example 2 *A commonly encountered nonlinear term is sin(x). For x "near" zero, $\sin(s) \approx x$ according to Taylor's series. If we need more accuracy we can keep the first two terms in the Taylor series expansion, but then we have a nonlinear system again. A comparison of sin(x) and its one and two term Taylor series expansions is shown in Figure 4. Clearly, the linear approximation of sin(x) rapidly becomes inaccurate for large magnitudes of x.*

3.1 Linearizing Nonlinear Differential Equations

As mentioned earlier, it may be impossible to analytically solve a nonlinear ODE, and it is impossible to control a nonlinear system using the methods covered in this course. However, we can linearize the nonlinear terms in a nonlinear system, and obtain an approximate solution which will be valid for small excursions of the dependent variable about an operating point.

Definition 4 *A differential equation is said to be nonlinear if any of its terms are nonlinear in the dependent variable, or any of the derivatives of the dependent variable with respect to the independent variable.*

Definition 5 *A system is said to be in a state of equilibrium if and only if its dependent variable is invariant with respect to time. In other words, all terms which are explicit functions of time, and all terms which are time-derivatives of the dependent variable must vanish.*

The dependent variable may be position, or velocity in dynamical systems. For example, a pendulum is known to be in equilibrium when in its vertical position. A sky diver is also in equilibrium when he or she reaches terminal velocity.

Thus, to locate the equilibrium point(s) of a system, set all time-derivatives of the dependent variable to zero, and ignore all explicit functions of time. Then, solve the resulting nonlinear equation for the equilibrium value of the dependent variable. Once the equilibrium point has been determined, the nonlinear terms may then be linearized about this point, and substituted back into the original equation of motion. If the equation of motion is linearized about an equilibrium point, any constant forcing terms will cancel out. If the equation of motion is linearized about any other point, there will still be some constant forcing terms in the equation. This makes sense if you realize that added force is required to hold a mechanism in any position other than an equilibrium position.

Example 3 *A mass of 1 kg is suspended by a nonlinear spring and damper as shown above. The tension force resisting the downward pull of gravity in the spring and damper is given by $F_T = -\left(2\dot{x}^3 + e^{2x}\right)$ Newtons, where x is measured from the unstretched length of the spring. The mass is subjected to an impulsive force of $10\delta(t)$ Newtons as shown. Find: (a) the nonlinear equation of motion, (b) the equilibrium point $\bar{x}$, (c) the linearized equation of motion in terms of the incremental deviation, $\hat{x}$, from the equilibrium point, and (d) the solution to the linearized equation of motion if $\hat{x}(0) = \dot{\hat{x}}(0) = 0$.*

Solution:

The nonlinear equation of motion is given by

$$\ddot{x} + 2\dot{x}^3 + e^{2x} = 10 + 10\delta\left(t\right) \tag{29}$$

At equilibrium, we have $\dot{x} = \ddot{x} = 0$, thus the equilibrium equation is given by
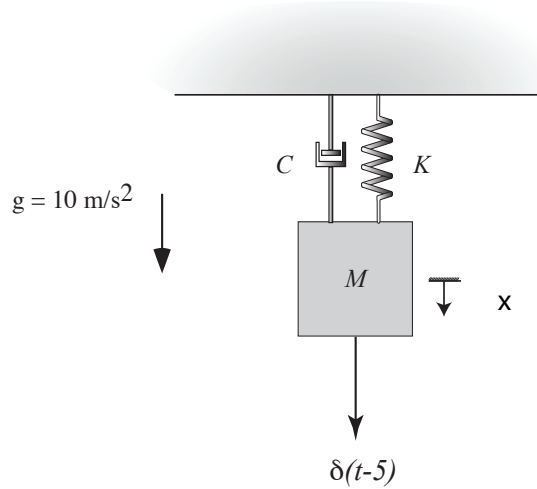
$$e^{2\bar{x}} = 10 \tag{30}$$

Figure 5. Nonlinear example.

Hence the equilibrium point is given by

$$\bar{x} = \frac{1}{2}\ln(10) \tag{31}$$

Linearizing, we obtain:

$$2\dot{x}^3 \approx 2\dot{\bar{x}}^3 + 6\dot{\bar{x}}^2\left(\dot{x} - \dot{\bar{x}}\right) = 0 \tag{32}$$

since $\dot{\bar{x}} = 0$, and

$$e^{2x} \approx e^{2\bar{x}} + 2e^{2\bar{x}}\left(x - \bar{x}\right) = 10 + 20\hat{x} \tag{33}$$

Substitution of the above into the original nonlinear equation of motion, and noting that $x = \bar{x} + \hat{x}(t)$, yields the *linearized EOM*[2]

$$\ddot{\hat{x}} + 20\hat{x} = 10\delta(t) \tag{34}$$

In the Laplace domain, we have

$$\hat{X}(s) = \frac{10}{s^2 + 20} \tag{35}$$

Hence, taking the inverse Laplace transform of Equation (35) yields

$$\hat{x}(t) = \frac{10}{\sqrt{20}}\sin\sqrt{20}tH(t) \tag{36}$$

It should be noted that the above is a rather crude model since it predicts a purely sinusoidal response by totally neglecting the nonlinear loss term.

4 Using Maple$^{\text{TM}}$ To Solve ODEs In The Laplace Domain

The symbolic mathematics program Maple is available for UNIX/Linux, Macintosh OSX, and Windows. Maple has an extensive on-line help facility which may be accessed at the command prompt by

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[2]A better linear approximation for the cubic damping may be developed using the concept of *equivalent viscous damping*. This is done by equating the energy dissipated over one complete cycle of harmonic motion by the nonlinear damper to that of a linear damper, and then solving for the required linear equivalent viscous damping constant.

entering ?subject. You are encouraged to explore the capabilities of Maple beyond what is required in ME307, but for now, we shall focus on a relatively small subset of Maple's capabilities. Specifically, we will use Maple to invert Laplace transforms, and plot the resulting time-domain solution. The following example will illustrate this process, but you will have to do some browsing through Maple's functions via the on-line help before you become comfortable with Maple's syntax.

Example 4 *Determine the unit step response for the system with transfer function given by*

$$T(s) = \frac{10\,(s + 10)}{s^2 + 6s + 25}$$

Solution:

The following commands are entered at the command line within Maple. First, Maple is executed (on UNIX systems) by entering "xmaple" at the prompt. Comments (non executed lines) are denoted by #. Suppose we forgot what the inverse Laplace command does, or that it requires us first to issue the readlib(laplace) command. ?invlaplace tells us what we need. The following series of commands, each of which is followed by a semicolon except the ? command, will be echoed to the screen (not shown here) by Maple. The end result is the plot shown below.

```
>?invlaplace

>with(inttrans);# Loads the integral transform library.
>T:=10*(s+10)/(s^2+6*s+25);
>R:=1/s;
>c:=invlaplace(R*T,s,t);
>plot(c,t=0..10);
```
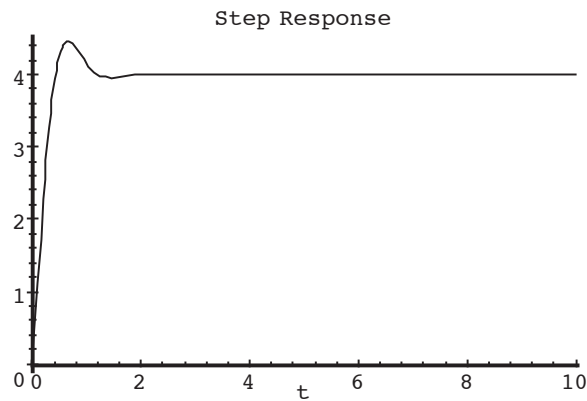

Figure 6. Maple plot.

There are several key commands with which one must become familiar to navigate Maple. Issue the following commands to learn about some of these functions: ?with, ?readlib, ?allvalues, ?solve, ?fsolve, ?dsolve, ?int, ?diff, ?simplify, ?expand, ?plot, ?odeplot, ?laplace, ?invlaplace. For example, issuing the ?Heaviside command yields the following output: FUNCTION: Dirac - the Dirac delta function FUNCTION: Heaviside - the Heaviside step function CALLING SEQUENCE: Dirac(t) Dirac(n, t) Heaviside(t) PARAMETERS:

t - algebraic expression
n - nonnegative integer
SYNOPSIS:
-The Dirac(t) delta function is defined as zero everywhere except at $t = 0$
where it has a singularity. It has an additional property, specifically:
Int(Dirac(t),t = -infinity..infinity) = 1
-Derivatives of the Dirac function are denoted by the two-argument Dirac func-
tion. The first argument denotes the order of the derivative.
For example, diff(Dirac(t), t$n) will
be automatically simplified to Dirac(n, t) for any integer n.
-The Heaviside(t) unit step function is defined as zero for t < 0, 1 for
t >= 0. It is related to the Dirac function by
diff(Heaviside(t),t) = Dirac(t).
-These functions are typically used in the context of integral transforms such
as laplace(), mellin() or fourier().
SEE ALSO: laplace, mellin, fourier