Method of Undetermined Coefficients

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The method of undetermined coefficients is used to determine the particular solution of *linear ordinary differential equations with constant coefficients when the forcing function has a finite number of linearly independent derivatives.*

For example, given an ODE of the form:

$$\ddot{x} + \alpha \dot{x} + \beta x = f(t), \tag{1}$$

where α and β are constants, we seek particular or forced solutions *proportional* to $f(t)$ and *all* of its derivatives – hence the need for a finite number of them!

Table 1. Forcing function and matching solution form.

Forcing Function	**Particular Solution**
$2t^3$	$At^3 + Bt^2 + C$
$10\sin(3t)$	$A\cos(3t) + B\sin(3t)$
$4e^{at}$	Ae^{at}
2	A

Luckily, the forcing function and solution pairs shown in Table 1 are fairly commonly encountered. However, the vast majority of possible functions do not have a finite number of linearly independent derivatives. For example:

$$\begin{aligned} cf(t) &= \frac{1}{t} \\ \dot{f}(t) &= -\frac{1}{t^2} \\ \ddot{f}(t) &= \frac{2}{t^3} \\ &\vdots \\ f^{(n)}(t) &= (-1)^n \frac{n!}{t^{n+1}} \end{aligned} \tag{2}$$

Another complication arises when the forcing function is proportional to a homogeneous solution. For example:

$$\ddot{x} - 4x = 3e^{-2t} \tag{3}$$

The homogeneous solutions to Equation 3 are Ae^{2t} and Be^{-2t}, so we must make the particular solution independent of the homogeneous solutions. This is accomplished by multiplying the repeated eigenfunction (homogeneous solution) by the independent variable, t. For the above example, the particular solution is given by:

$$x_p(t) = -\frac{3}{4} t e^{-2t} \tag{4}$$

A systematic approach to dealing with the above situation is found in the so-called annihilator operator theory – a discussion of which may be found in any text on elementary ordinary differential equations.