ME 279 Supplemental Notes for Solving Ordinary Differential Equations Via the Method of Undetermined Coefficients – with Maple™ Examples

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For linear ODE's with constant coefficients, there are three main methods of solving for the solutions:

1. Undetermined coefficients-the particular solution – usually the easiest when applicable
2. Variation of parameters-complicated except for first-order ODEs
3. Laplace transformations (or other integral methods)
   - use when step functions or delta functions are present in the forcing term (rhs of the equation).
   - use when method of undetermined coefficients won't work.

When does method of Undetermined Coefficients work?

* When the forcing term (on the rhs of the equation) has a finite # of linearly independent derivatives.

Forcing	Trial Solution
t^2	$At^2 + Bt + C$
$\sin \omega t$	$A \sin \omega t + B \cos \omega t$
10 cos ωt	$A \sin \omega t + B \cos \omega t$
$20e^{at}$	Ae^{at} (watch out for forcing by eigen-functions though)
10	A

When won't method of Undetermined Coefficients work?

* ***When the forcing term has an infinite number of linearly independent derivatives***. For example: *f(t)=1/t* *or* *f(t)= ln(t)*.

Homogeneous Solutions for linear ODE's with constant coefficients

An n^th order homogeneous linear ODE with constant coefficients of the form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = 0 , \qquad (1)$$

always has a solution of the form:

$$Ae^{\lambda t} \tag{2}$$

Substitution of (2) into (1) yields the characteristic polynomial equation:

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0 = 0\,. \tag{3}$$

The characteristic polynomial in Equation (3) can ***always*** be factored into products of first and second-order factors. For example:

$$\begin{aligned} &\lambda^3 + 20\,\lambda^2 + 108\,\lambda + 80 \\ &= (\lambda + 10)\,\left(\lambda^2 + 10\,\lambda + 8\right) \end{aligned}$$

which has roots: $-10, -5 + j\sqrt{17}, -5 - j\sqrt{17}$.

As a consequence of the fact that in most ODEs the coefficients, a_i, are Real, the characteristic polynomial may have only real roots or complex conjugate roots. Complex eigenvalues lead to complex exponential solutions but may always be recast in terms of sines and cosines via Euler's famous relationship:

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta \tag{4}$$

Hence, the general form of the solution to the homogeneous ODE given by

$$\dddot{x} + 20\ddot{x} + 108\dot{x} + 80x = 0 \tag{5}$$

may be written:

$$x(t) = A_1 e^{-10t} + e^{-5t}\left(B_2 \cos\sqrt{17}t + B_2 \sin\sqrt{17}t\right) \tag{6}$$

In (6), B_1 and B_2 have been used to emphasize that the constants A_i and B_i are different when switching from the exponential form to the harmonic (containing sines and cosines) form. How are they related?

Some Examples:

a) For the differential equation,

$$\ddot{y} + 9y = \sin t$$

with the initial conditions

$$y(0) = 1,$$

and,

$$\dot{y}(0) = -1,$$

the solution may be found to be

$$y(t) = -\frac{3}{8}\sin(3\,t) + \cos(3\,t) + \frac{1}{8}\sin(t)$$

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The Maple commands used to generate this answer are:

```
> eqn:=diff(y(t),t$2)+9*y(t)-sin(t);
```

$$eqn := \left(\frac{\partial^2}{\partial t^2}\,y(t)\right) + 9\,y(t) - \sin(t)$$

```
> dsolve({eqn=0,y(0)=1,D(y)(0)=-1},y(t));
```

$$y(t) = -\frac{3}{8}\sin(3\,t) + \cos(3\,t) + \frac{1}{8}\sin(t)$$

b) The (homogeneous) differential equation,

$$\ddot{y} + \pi^2 y = 0$$

with initial conditions

$$y(0) = 2$$

and

$$\dot{y}(0) = -2$$

has solution:

```
> eqn2:=diff(y(t),t$2)+Pi^2*y(t);
```

$$eqn2 := \left(\frac{\partial^2}{\partial t^2}\,y(t)\right) + \pi^2\,y(t)$$

```
> dsolve({eqn2=0,y(0)=2, D(y)(0)=-2},y(t));
```

$$y(t) = -2\,\frac{\sin(\pi\,t)}{\pi} + 2\cos(\pi\,t)$$

The above solutions were obtained using Maple version 10. For more information, please see:

1. Dr. Stutts' Maple Vault: http://web.umr.edu/~stutts/MapleVault.htm
2. Maplesoft, Inc. Website: http://www.maplesoft.com/