The Phasor Analysis Method For Harmonically Forced Linear Systems

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1 Introduction

One of the most common tasks in vibration analysis is the determination of the steady-state response of a system to harmonic forcing. This is also true in AC electrical circuit analysis. The most common case encountered in vibration engineering is that of the forced underdamped system. In this case, the transient portion of the response vanishes exponentially as $e^{-\zeta\omega t}$, where ζ is the so-called *dimensionless damping constant*, and for the underdamped case, $0 < \zeta < 1$. The prototypical mass-spring-damper system is shown in Figure 1, where in the harmonically forced case, we shall assume that $F(t) = F_0 \sin \omega t$, where ω is the forcing frequency in $\frac{radians}{second}$.

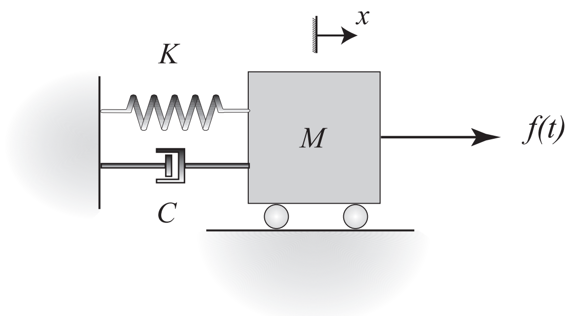

Figure 1. Prototypical forced mass-spring-damper system

In the following, we shall determine the steady-state response to harmonic forcing via two methods: (1) the conventional approach via the *method of undetermined coefficients*, and (2) the *Phasor method*. First, we note that the equation of motion is given by

$$M\ddot{x} + C\dot{x} + Kx = F_0 \sin \omega t. \tag{1}$$

Equation (1) may be recast as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{M} \sin \omega t. \tag{2}$$

Equation (2) is the so-called *canonical* form of the equation of motion. Assuming the underdamped case ($0 < \zeta < 1$), we may find the *homogeneous* or *transient* solution is given by

$$x_h(t) = e^{-\zeta\omega_n t} \left(A \cos \omega_d t + B \sin \omega_d t\right), \tag{3}$$

where the *undamped natural frequency* $\omega_n = \sqrt{\frac{K}{M}}$, and the *damped natural frequency* $\omega_d = \omega_n\sqrt{1-\zeta^2}$

2 The Forced Response Via the Method of Undetermined Coefficients

The forced or *particular* solution is given by

$$x_p(t) = C_1 \cos \omega t + C_2 \sin \omega t. \tag{4}$$

Substitution of Equation (4) into Equation (2) yields

$$\begin{aligned}\left(-\omega^2 C_1 \cos \omega t - \omega^2 C_2 \sin \omega t\right) + 2\zeta\omega_n\omega\left(-C_1 \sin \omega t + C_2 \cos \omega t\right)\\ +\omega_n^2\left(C_1 \cos \omega t + C_2 \sin \omega t\right) = \frac{F_0}{M}\sin \omega t.\end{aligned} \tag{5}$$

Collecting the coefficients of sin and cosine in Equation (6) yields

$$-2\zeta\omega_n\omega C_1 + \left(\omega_n^2 - \omega^2\right) C_2 = \frac{F_0}{M}, \tag{6}$$

and

$$\left(\omega_n^2 - \omega^2\right) C_1 + 2\zeta\omega_n\omega C_2 = 0. \tag{7}$$

Dividing Equations (6) and (7) by ω_n^2, and recasting them in matrix form yields

$$\begin{bmatrix} -2\zeta r & \left(1-r^2\right) \\ \left(1-r^2\right) & 2\zeta r \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \frac{F_0}{K} \\ 0 \end{pmatrix} \tag{8}$$

where $r = \frac{\omega}{\omega_n}$. Solving for C_1 and C_2 yields

$$C_1 = -\frac{2\zeta\, r \frac{F_0}{K}}{\left(2\zeta r\right)^2 + \left(1-r^2\right)^2}, \tag{9}$$

and

$$C_2 = \frac{\left(1-r^2\right)\frac{F_0}{K}}{\left(2\zeta r\right)^2 + \left(1-r^2\right)^2}. \tag{10}$$

Hence, the total solution is given by

$$\begin{aligned}x(t) = \frac{F_0}{KR^2}\left(-2\zeta r \cos \omega t + \left(1-r^2\right)\sin \omega t\right)\\ +e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B \sin \omega_d t\right),\end{aligned} \tag{11}$$

where A and B depend on the initial conditions, and

$$R^2 = (2\zeta r)^2 + (1-r^2)^2. \tag{12}$$

Clearly, as $t \to \infty$, $x(t) \to x_p(t)$, where, $x_p(t)$ is the particular, or steady-state solution to Equation (2).

Thus, at steady-state, $x(t) = x_p(t)$, and we have

$$x(t) = \frac{F_0}{KR}\left(\sin\phi\cos\omega t - \cos\phi \sin\omega t\right), \tag{13}$$

where,

$$\frac{2\zeta r}{R} = \sin\phi, \tag{14}$$

and

$$\frac{1-r^2}{R} = \cos\phi. \tag{15}$$

From trigonometry[1], we have $\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$. Hence, we may rewrite Equation (13) as

$$x(t) = \frac{F_0}{KR}\sin(\omega t - \phi), \tag{16}$$

[1]See Equation (7) in "Some Useful Complex Algebra and Trigonometric Relationships," D. S. Stutts, March 10, 2004.

where,

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{2\zeta r}{1-r^2}. \tag{17}$$

Thus,

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \text{ for } r \leq 1, \tag{18}$$

or

$$\phi = \pi + \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \text{ for } r > 1. \tag{19}$$

Equation (16) reveals that the steady-state response of the system may be written in terms of a magnitude and a phase shift.

$$x(t) = X\sin\left(\omega t - \phi\right), \tag{20}$$

where

$$X = \frac{F_0}{KR} = \frac{F_0}{K\sqrt{\left(2\zeta r\right)^2 + \left(1-r^2\right)^2}}. \tag{21}$$

Examination of Equation (20) leads to a *much* quicker route of analysis.

From Equations (13) through (17), we may write the displacement X as a complex vector in the following manner

$$\mathbf{X} = \frac{F_0}{KR}\left[\cos\phi - j\sin\phi\right]. \tag{22}$$

Let us now define a mechanical impedance $Z(r)(=Z(\omega))$ – analogous to electrical impedance except that in this case, *current and voltage are replaced by force and displacement* respectively – given by
Hence,

$$\mathbf{Z}(r) = \frac{\mathbf{X}}{F_0} = \frac{1}{KR}\left[\cos\phi - j\sin\phi\right], \tag{23}$$

so

$$\mathbf{X} = \mathbf{Z}(r)F_0. \tag{24}$$

From Equations (14) and (15), the impedance may be recast in polar form (magnitude and phase) as

$$\mathbf{Z}(r) = \frac{\mathbf{X}}{F_0} = \frac{e^{-j\phi}}{K\sqrt{\left(2\zeta r\right)^2 + \left(1-r^2\right)^2}} = \frac{e^{-j\phi}}{KR}. \tag{25}$$

In equation (24), F_0 is the magnitude of the harmonic input force, and $\mathbf{X}$ provides information about the magnitude as well as the direction of the response. Defining the real and imaginary parts of $\mathbf{Z}$ as

$$R_1 = \frac{\cos\theta}{KR} \tag{26}$$

and

$$R_2 = \frac{\sin\theta}{KR} \tag{27}$$

the mechanical impedance for the system is shown graphically in Figure 2[2].

If we consider a complex forcing function given by

$$\mathbf{f}(t) = F_0 e^{j\omega t} = F_0\left(\cos\omega t + j\sin\omega t\right), \tag{28}$$

Then, we may compute the response $\mathbf{r}(t)$ from Equations (23) and (24)

$$\mathbf{r}(t) = \mathbf{Z}(r)\mathbf{f}(t) = \frac{F_0}{KR}\left[\cos\phi - j\sin\phi\right]\left(\cos\omega t + j\sin\omega t\right). \tag{29}$$

[2] Although the the phase is actually negative, it is shown here in the first quadrant for clarity.

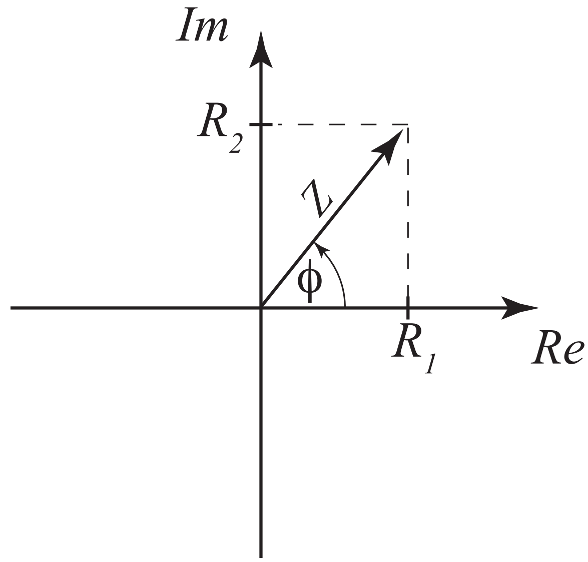

Figure 2. Polar (phasor) representation of the mechanical impedance in the complex plane.

Completing the multiplication in Equation (29), and grouping real and imaginary terms yields

$$\mathbf{r}(t) = \frac{F_0}{KR}\left[\cos\phi\cos\omega t + \sin\phi\sin\omega t + j\left(\cos\phi\cos\omega t - \sin\phi\sin\omega t\right)\right]. \tag{30}$$

or

$$\mathbf{r}(t) = \frac{F_0}{KR}\cos\left(\omega t - \phi\right) + j\frac{F_0}{KR}\sin\left(\omega t - \phi\right). \tag{31}$$

Defining two complex vector operations $Re\left\{\mathbf{r}(t)\right\}$, and $Im\left\{\mathbf{r}(t)\right\}$ to take the real and imaginary parts of $\mathbf{r}(t)$ respectively, Equation (31) may be written as

$$\mathbf{r}(t) = \frac{F_0}{KR}\left[Re\left\{e^{(\omega t - \phi)}\right\} + jIm\left\{e^{(\omega t - \phi)}\right\}\right]. \tag{32}$$

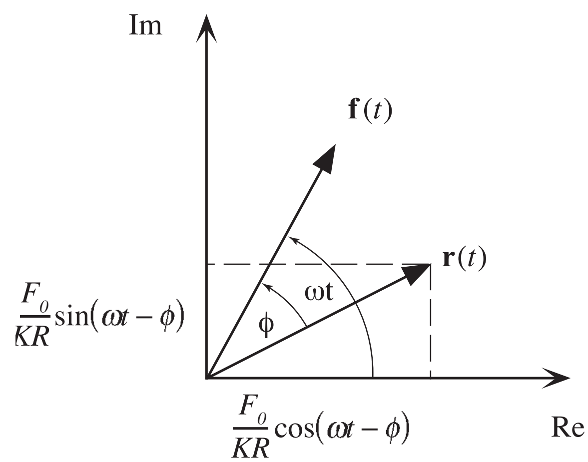

Figure 3. Polar (phasor) representation of the system response to harmonic forcing

Hence, we see that the steady-state system response is related to the harmonic forcing as shown in Figure 3.

Thus, the steady-state response of a system to a harmonic forcing function, $f(t) = F_0 \sin\omega t$, or $f(t) = F_0\cos\omega t$, may be computed by assuming $f(t) = F_0 e^{j\omega t}$, and taking the imaginary or real part of the result respectively. It is clear from Figure 3 that the steady-state response of a system to either $f(t) = F_0 \sin \omega t$, or $f(t) = F_0 \cos \omega t$ may be thought of as a projection of the response due to $f(t) = F_0 e^{j\omega t}$ on the imaginary or real axes in the complex plane.

3 The Phasor Method

Given the result shown in Equations (31) and (32), we may approach the solution of Equation (2) for the steady--state response in a completely different manner. Letting $x(t) = X\mathrm{Im}\left\{e^{j\omega t}\right\}$, and substituting into Equation (2), yields[3]

$$\left[\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega\right] X\mathrm{Im}\left\{e^{j\omega t}\right\} = \frac{F_0}{M} Im\left\{e^{j\omega t}\right\} = \frac{F_0\omega_n^2}{K} Im\left\{e^{j\omega t}\right\}. \tag{33}$$

Canceling the phasors on either side of Equation (33),dividing by ω_n^2, and solving for X, yields

$$X = \frac{F_0}{K\left[1 - r^2 + j2\zeta r\right]} = \frac{F_0 e^{-j\phi}}{K\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}}, \tag{34}$$

where,as before, $r = \frac{\omega}{\omega_n}$, and ϕ is given by Equation (18). Thus, the steady-state solution is given by

$$x(t) = \frac{F_0 e^{-j\phi}}{K\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}} \mathrm{Im}\left\{e^{j\omega t}\right\} = \frac{F_0}{K\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}} \mathrm{Im}\left\{e^{j(\omega t - \phi)}\right\}, \tag{35}$$

or,

$$x(t) = \frac{F_0}{K\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}} \sin\left(\omega t - \phi\right) = |X| \sin\left(\omega t - \phi\right). \tag{36}$$

Examination of Equations (12) and (21), reveals that the result given in Equation (36) is identical to that obtained in Equation (20), but using the phasor method, required far fewer algebraic steps. The approach is identical for $f(t) = F_0 \cos \omega t$, except that a solution of the form $x(t) = \mathrm{Re}\left\{e^{j\omega t}\right\}$ is assumed instead.

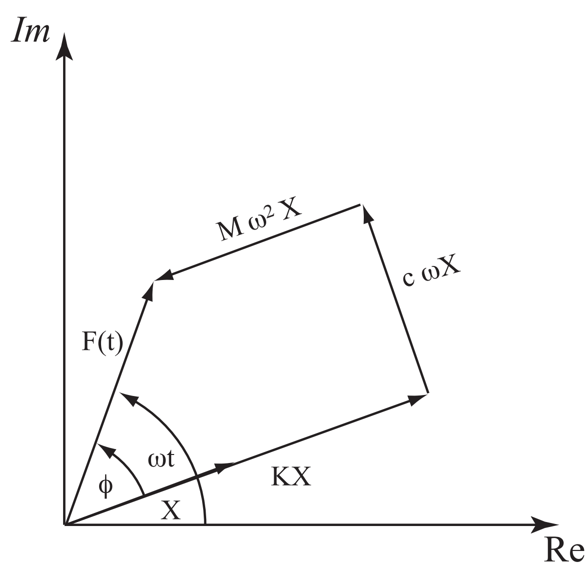

Figure 4. Force vector relationships for a harmonically forced spring-mass-damper system.

The phase relationship between the individual force components of the system as described by Equation (1) are shown graphically in Figure 4, where it may be seen that the damper force and inertial forces are 90° and 180° out of phase with the spring force respectively.

[3] The Real and Imaginary components yield two scalar equations.

The phasor method is credited to Charles Proteus Steinmetz who was an engineer at General Electric around the turn of the century. With the growing importance of AC circuitry, Steinmetz was compelled to develop a quicker method of analysis because determining the steady-state response of even relatively simple circuits to harmonic voltage excitation is *extremely* tedious using conventional methods.