Some Useful Complex Algebra and Trigonometric Relationships

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On the road to becoming a "card carrying wave mechanic," there are several very useful trigonometric relationships and properties which should be understood. One of the most basic is Euler's relationship

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta. \tag{1}$$

Equation (1) makes it very easy to derive many useful identities:

$$e^{j(\theta+\phi)} = \cos(\theta+\phi) + j\sin(\theta+\phi) = (\cos\theta + j\sin\theta)(\cos\phi + j\sin\phi), \tag{2}$$

and

$$e^{j(\theta-\phi)} = \cos(\theta-\phi) + j\sin(\theta-\phi) = (\cos\theta + j\sin\theta)(\cos\phi - j\sin\phi). \tag{3}$$

Equations (2) and (3) lead to the following additive and difference angle formulas:

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi, \tag{4}$$

$$\cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi, \tag{5}$$

$$\sin(\theta+\phi) = \cos\theta\sin\phi + \sin\theta\cos\phi, \tag{6}$$

$$\sin(\theta-\phi) = \sin\theta\cos\phi - \cos\theta\sin\phi. \tag{7}$$

Equations (4) – (7) lead to the following product identities:

$$\sin\theta\cos\phi = \frac{1}{2}\left\{\sin(\theta+\phi) + \sin(\theta-\phi)\right\}, \tag{8}$$

$$\sin\theta\sin\phi = \frac{1}{2}\left\{\cos(\theta-\phi) - \cos(\theta+\phi)\right\}, \tag{9}$$

$$\cos\theta\cos\phi = \frac{1}{2}\left\{\cos(\theta+\phi) + \cos(\theta-\phi)\right\}, \tag{10}$$

Other identities useful in describing the so-called *beating phenomenon* are:

$$\sin\theta \pm \sin\phi = 2\sin\left(\frac{\theta\pm\phi}{2}\right)\cos\left(\frac{\theta\mp\phi}{2}\right), \tag{11}$$

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right), \tag{12}$$

and

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right) \tag{13}$$

Equations (4) – (7) have another useful application in describing the sum of a cosine and a sine term of the same frequency, but different amplitudes. For example: let $z(t) = A\cos\omega t + B\sin\omega t$. Then, we may write

$$z(t) = \sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}}\cos\omega t + \frac{B}{\sqrt{A^2+B^2}}\sin\omega t\right]. \tag{14}$$

We may then apply, for example, Equation (5) to obtain

$$z(t) = \sqrt{A^2+B^2}\cos(\omega t - \phi), \tag{15}$$

where,

$$\phi = \tan^{-1}\left(\frac{B}{A}\right). \tag{16}$$

Note that Equation (14) may be expressed as

$$z(t) = \sqrt{A^2+B^2}\,Re\left\{e^{j(\omega t-\phi)}\right\}, \tag{17}$$

where *Re* denotes the Real part of Equation (1). Hence, Equation (17) may be thought of as the Real projection of a vector in the complex plane. This is called the *phasor* representation of a harmonic function, and is used extensively in the study of steady-state harmonic vibrations as well as AC circuit analysis.

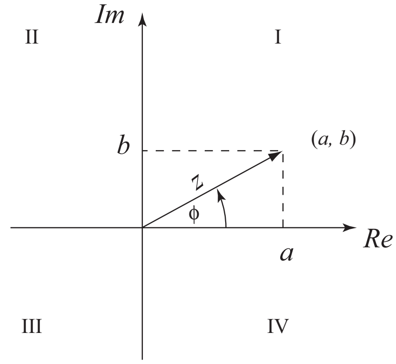

Figure 1. Graphical representation of a complex number.

Any complex number may be represented in either Cartesian or polar form. For example:

$$z = a + jb = \sqrt{a^2 + b^2} e^{j\phi} \tag{18}$$

where $a = Re\{z\}$, and $b = Im\{z\}$, and a and b may be positive or negative, the phase may be written

$$\phi = \begin{cases} \tan^{-1}\left(\frac{b}{a}\right) & \text{for } z \text{ in Quad. I} \\ 180^\circ + \tan^{-1}\left(\frac{b}{a}\right) & \text{for } z \text{ in Quad. II and III} \\ 360^\circ + \tan^{-1}\left(\frac{b}{a}\right) & \text{for } z \text{ in Quad. IV} \end{cases} \tag{19}$$

The relationship between Cartesian and polar forms may be seen graphically in Figure 1, where $z = re^{j\phi}$, so

$$r = \sqrt{a^2 + b^2}, \tag{20}$$

$$a = r \cos\phi, \tag{21}$$

and

$$b = r \sin\phi. \tag{22}$$

Polar form is often more convenient for calculations. For example, let $z_1 = a + jb$, and $z_2 = c + jd$, we have the following relations:

$$\frac{z_1}{z_2} = \frac{a + jb}{c + jd} = \frac{a + jb}{c + jd}\left(\frac{a - jb}{c - jd}\right) = \frac{ac + bd}{c^2 + d^2} + j\frac{bc - ad}{c^2 + d^2}, \tag{23}$$

but in polar form, we have

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\phi_1}}{r_2 e^{\phi_2}} = \frac{r_1}{r_2} e^{j(\phi_1 - \phi_2)}, \tag{24}$$

which is computationally simpler.

Polar form is especially useful in extracting roots of numbers. for example, the cube root of 27 may be found as follows:

$$z^3 - 27 = 0 \Rightarrow z^3 = 27 = 27e^{j2n\pi}, \tag{25}$$

where n is an integer. Hence, we have

$$n = 1 \Rightarrow z = 3e^{j2\pi/3} = -\frac{3}{2} + j\frac{3}{2}\sqrt{3}, \tag{26}$$

$$n = 2 \Rightarrow z = 3e^{j4\pi/3} = -\frac{3}{2} - j\frac{3}{2}\sqrt{3}, \tag{27}$$

and

$$n = 3 \Rightarrow z = 3e^{j2\pi} = 3. \tag{28}$$

Note that the roots repeat, in this case, for $n > 3$. In general, we have[1]

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{j\frac{\phi}{n}} = r^{\frac{1}{n}} \left(\cos\frac{\phi}{n} + j \sin\frac{\phi}{n} \right). \tag{29}$$

Similarly, we have

$$z^n = r^n \, e^{j\, n\phi} = r^n \left(\cos n\phi + j \sin n\phi \right). \tag{30}$$

For example, from the above, it may be shown that

$$\cos^3 \theta = \frac{1}{4} \left(3 \cos\, \theta + \cos 3\theta \right) \tag{31}$$

and

$$\sin^3 \theta = \frac{1}{4} \left(3 \sin\, \theta - \sin 3\theta \right) \tag{32}$$

Similarly,

$$\cos^4 \theta = \frac{1}{8} \left(3 + 4 \cos\, 2\theta + \cos 4\theta \right) \tag{33}$$

and

$$\sin^4 \theta = \frac{1}{8} \left(3 - 4 \cos\, 2\theta + \cos 4\theta \right) \tag{34}$$

[1] This is an application of DeMoivre's identity. See: `http://en.wikipedia.org/wiki/De_Moivre_formula`.