AE/ME 339

Computational Fluid Dynamics (CFD)

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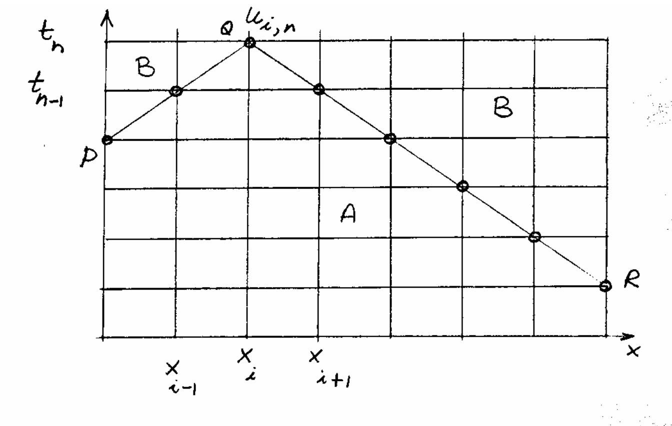
Implicit form of difference equation

In the previous explicit method, the solution at time level n, u i,n , depended only on the known values of u, u i-1,n-1 , u i,n-1, and u i+1,n-1 , all which are at time level n-1.

Note: $u_{i,n}$ and u_i^n mean the same.

We will be using these interchangeably.

Nature of solution in explicit method can be illustrated graphically as shown below.



In this formulation, solution at (i,n), $u_{i,n}$ is affected only by the values along and below boundary PQR (region A) in the previous figure.Values in region above PQR (region B) do not influence $u_{i,n}$.

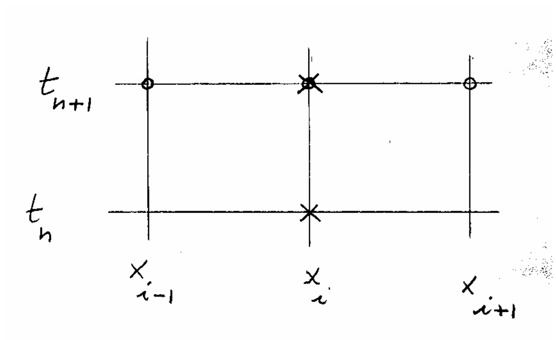
Exact solution u(x,y) at Q depends on the values at all times earlier than t_n, a property of parabolic PDE.

Is a limitation of the explicit method.

CFL criterion for stability: $0 \le \dfrac{\Delta t}{(\Delta x)^2} \le \dfrac{1}{2}$

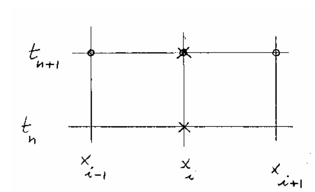
Fully Implicit Method

The nodes used in implicit method are illustrated in the figure below



As before, crosses (x) denote grid points used for $\frac{\partial u}{\partial t}(u_t)$ and circles (o) for $\frac{\partial^2 u}{\partial x^2}(u_{xx})$

The equation now becomes

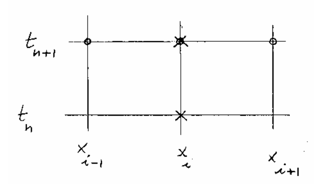


$$\frac{u_{i,n+1} - u_{i,n}}{\Delta t} = \frac{u_{i-1,n+1} - 2u_{i,n+1} + u_{i+1,n+1}}{(\Delta x)^2}$$

[illegible]

[illegible]

IC and BC are the same as before.



$$u_{0,n+1} = g_0(n+1)$$

$$u_{M,n+1} = g_1(n+1)$$

$$u_{i,0} = f(x_i)$$

Equations similar to the above should be written for each grid point

$$1 \le i \le M - 1$$

Note: left boundary has i=0 and right boundary has i=M, thus total of (M+1) grid points (also knows as nodes) are present.
Thus we have (M-1) linear simultaneous equations with (M-1) unknowns.
Explicit solution is not possible.

<u>Convergence of Implicit form</u>

Can show using Taylor series

$$u_{i,n+1} - u_{i,n} = \lambda u_{i-1,n+1} - 2\lambda u_{i,n+1} + \lambda u_{i+1,n+1} + z_{i,n}$$

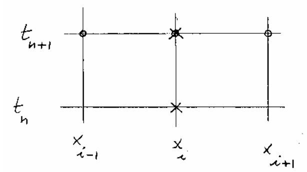
Where

$$\frac{z_{i,n}}{\Delta t} = -\left[\frac{\Delta t}{2} u_{tt} + \frac{(\Delta x)^2}{12} u_{xxxx}\right] + O\left[(\Delta t)^2\right] + O\left[(\Delta x)^4\right]$$

u_{tt} and u_{xxxx} are evaluated at (i , n+1).

From the leading in the above

$$\frac{z_{i,n}}{\Delta t} = O\left[\Delta t + (\Delta x)^2\right]$$



It can be shown that implicit method converges to the exact solution of the PDE as $\Delta t \to 0$ and $\Delta x \to 0$ for any value of $\frac{\Delta t}{(\Delta x)^2}$

The difference equation is now written for $1 \le i \le M-1$ as follows.

$$(1+2\lambda)u_{1,n+1} - \lambda u_{2,n+1} = u_{1,n} + \lambda g_0(t_{n+1}) \qquad \text{for } i = 1$$

$$-\lambda u_{i-1,n+1} + (1+2\lambda)u_{i,n+1} - \lambda u_{i+1,n+1} = u_{i,n} \qquad \text{for } 2 \le i \le M-2$$

Finally, (show M, N in figure)

$$-\lambda u_{M-2,n+1} + (1+2\lambda)u_{M-1,n+1} = u_{M-1,n} + \lambda g_1(t_{n+1})$$

for i = M

$$\begin{bmatrix} b_1 & c_1 & & & & & \\ a_2 & b_2 & c_2 & & & & \\ & a_3 & b_3 & c_3 & & & \\ & & & a_i & b_i & c_i & \\ & & & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & & & a_N & b_N \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_i \\ u_{N-1} \\ u_N \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_i \\ d_{N-1} \\ d_N \end{Bmatrix}$$

Note: In the matrix shown above, change in subscript N=M-1 is adopted for simplicity.

The RHS terms $d_1, d_2, \ldots d_N$ are known quantities. All matrix elements not shown are zero.
The matrix above is called a tridiagonal matrix i.e., only sub-diagonal, diagonal, and super diagonal terms are non–zero.

Solution can be obtained by Gauss elimination.

Recursion solution

$$u_i = \gamma_i - \frac{c_i}{\beta_i} u_{i+1}$$

Constants βi and γi are to be determined.

Computational Fluid Dynamics (AE/ME 339) K. M. Isaac
MAEEM Dept., UMR

Substituting into the i^{th} equation of the set for u_{i-1} gives

$$a_i \left(\gamma_{i-1} - \frac{c_{i-1}}{\beta_{i-1}} u_i \right) + b_i u_i + c_i u_{i+1} = d_i$$

Rewrite as

$$u_i = \frac{d_i - a_i \gamma_{i-1}}{b_i - \frac{a_i c_{i-1}}{\beta_{i-1}}} - \frac{c_i u_{i+1}}{b_i - \frac{a_i c_{i-1}}{\beta_{i-1}}}$$

$$u_i = \gamma_i - \frac{c_i}{\beta_i} u_{i+1}$$

Comparing the two equations we have recursion relations for β and γ

$$\beta_i = b_i - \frac{a_i c_{i-1}}{\beta_{i-1}} \qquad u_i = \gamma_i - \frac{c_i}{\beta_i} u_{i+1}$$

$$\gamma_i = \frac{d_i - a_i \gamma_{i-1}}{\beta_i}$$

From the first equation (see matrix in slide 10)

$$u_1 = \frac{d_1}{b_1} - \frac{c_1}{b_1} u_2$$

Where $\beta_1 = b_1$ and $\gamma_1 = \dfrac{d_1}{\beta_1}$

$$u_i = \gamma_i - \frac{c_i}{\beta_i} u_{i+1}$$

From the last equation (see matrix in slide 10)

$$u_N = \frac{d_N - a_N u_{N-1}}{b_N} = \frac{d_N - a_N \left(\gamma_{N-1} - \frac{c_{N-1}}{\beta_{N-1}} u_N \right)}{b_N}$$

Rearranging yields

$$u_i = \gamma_i - \frac{c_i}{\beta_i} u_{i+1}$$

$$u_N = \frac{d_N - a_N \gamma_{N-1}}{b_N - \frac{a_N c_{N-1}}{\beta_{N-1}}} = \gamma_N$$

Algorithm summary:

Recursion formulas for β_i and γ_i

$$\beta_1 = b_1, \ \gamma_1 = \frac{d_1}{\beta_1}$$

$$\beta_i = b_i - \frac{a_i c_{i-1}}{\beta_{i-1}}, \qquad i = 2, 3, \ldots, N$$

$$\gamma_i = \frac{d_i - a_i \gamma_{i-1}}{\beta_i}, \qquad i = 1, 2, 3, \ldots, N$$

Once the coefficients have been calculated, the solution vector u can be calculated starting with uN and going backwards, as follows:

$$u_N = \gamma_N$$

$$u_i = \gamma_i - \frac{c_i u_{i+1}}{\beta_i}, \qquad \text{i=N-1, N-2,…,1}$$

Note: n denotes time level and N denotes the last but one node in the i-direction.

Recursion formulas for β_i and γ_i

$$\beta_1 = b_1,\ \gamma_1 = \frac{d_1}{\beta_1}$$

$$\beta_i = b_i - \frac{a_i c_{i-1}}{\beta_{i-1}}, \qquad \text{i = 2, 3,…,N}$$

$$\gamma_i = \frac{d_i - a_i \gamma_{i-1}}{\beta_i}, \qquad \text{i = 1, 2, 3,…,N}$$

Gauss elimination can cause large round-off errors.
Implicit schemes usually require more computational steps, but the ratio Δt/(Δx)2 has no restrictions, is a definite advantage.

Stability (7.10)

A finite difference form is convergent if the solution tends to the exact solution as $(\Delta t, \Delta x) \to 0$ (in the absence of round off error).

Stability refers to amplification of information present in IC, BC or introduced by errors in the numerical procedure such as round off error.

Von-Neumann's stability analysis:
Stability implies only boundedness, not the magnitude of deviation from the true solution.

Key features of stability analysis:
Assume that: i) At any stage, t=0 here, a Fourier expansion can be made of some initial function f(x), and a typical term in the expansion can be written as $e^{j\beta x}$ where β is a positive constant and $j = \sqrt{-1}$

ii) Separation of time and space dependence can be made.

At time t, the term becomes $\psi(t)e^{j\beta x}$

By substituting in the difference equation, the form of ψ(t) can be determined and stability criterion established.

Example

Explicit finite difference form

$$\frac{u_{i,n+1} - u_{i,n}}{\Delta t} = \frac{u_{i-1,n} - 2u_{i,n} + u_{i+1,n}}{(\Delta x)^2}$$

Substitute for each u.

$$\frac{\psi(t+\Delta t)e^{j\beta x}-\psi(t)e^{j\beta x}}{\Delta t} = \frac{\psi(t)}{(\Delta x)^2}\left[e^{j\beta(x-\Delta x)}-2e^{j\beta x}+e^{j\beta(x+\Delta x)}\right]$$

Cancel exp (jβx) throughout

$$\psi(t + \Delta t) = \psi(t)\left[1+\lambda(e^{-j\beta\Delta x}-2+e^{j\beta\Delta x})\right]$$

Trig. identity: $Cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$ can be used to get

$$\psi(t + \Delta t) = \psi(t)\left[1+\lambda(-2+2\cos(\beta\Delta x))\right]$$

Since $\cos\theta = 1-2\sin^2(\theta/2)$

$$\psi(t + \Delta t) = \psi(t)\left[1-2\lambda\left(2\sin^2\frac{\beta\Delta x}{2}\right)\right]$$

$$\psi(t+\Delta t) = \psi(t)\left[1-4\lambda\sin^2\left(\frac{\beta\Delta x}{2}\right)\right]$$

If we choose $\psi(0) = 1$, this has the solution

$$\psi(t) = \left(1-4\lambda\sin^2(\beta\Delta x/2)\right)^{(t/\Delta t)}$$

And can be proven by substitution (see next slide).

$$\psi(t+\Delta t) = \left(1-4\lambda\sin^2(\beta\Delta x/2)\right)^{(t+\Delta t)/\Delta t}$$

$$= \psi(t)\left[1-4\lambda\sin^2(\beta\Delta x/2)\right]$$

For stability, ψ(t) must be bounded as $(\Delta t, \Delta x) \to 0$

This requires

$$\left|1-4\lambda\sin^2(\beta\Delta x/2)\right| \le 1$$

An amplification factor ξ is usually defined as follows:

Computational Fluid Dynamics (AE/ME 339) K. M. Isaac
MAEEM Dept., UMR

$$\xi = 1 - 4\lambda\sin^2(\beta\Delta x/2)$$

Which shows $|\xi| \le 1$ for stability.

In the Fourier expansion we considered only one term corresponding to 1 value of β. When all possible values of β are considered sin(βΔx/2) could become 1.

Therefore, the stability condition becomes

$$\lambda = \frac{\Delta t}{(\Delta x)^2} \le \frac{1}{2}$$

λ = 0.5, |ξ| = 1
λ = 1.0, |ξ| = 3 (unstable)

Intuitively it implies that $u_{i,n}$ affects $u_{i,n+1}$ in a “non-negative” manner.

A similar analysis for the implicit method would give

$$\xi = \frac{1}{1+4\lambda \sin^2(\beta\Delta x/2)}$$

Since $\xi \leq 1$ for all λ, the procedure is unconditionally stable.

Consistency means that the procedure may in fact approximate the solution of the PDE under study and not the solution of some other PDE.

Program Completed

University of Missouri-Rolla