AE/ME 339

Computational Fluid Dynamics (CFD)

K. M. Isaac

Governing equation summary

Continuity equation

Non-conservation form

$$\frac{D\rho}{Dt} + \rho\overline{\nabla}.\overline{V} = 0 \quad \ldots\ldots\ldots\ldots(2.29)$$

Conservation form

$$\frac{\partial\rho}{\partial t} + \overline{\nabla}\cdot(\rho\overline{V}) = 0 \ldots\ldots\ldots(2.33)$$

Governing equation summary

Momentum equation

Non-conservation form

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x ....(2.50a)$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y ....(2.50b)$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z ....(2.50c)$$

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Governing equation summary

Momentum equation

conservation form

$$\frac{\partial(\rho u)}{\partial t} + \bar{\nabla} \cdot \left(\rho u \bar{V}\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x ....(2.56a)$$

$$\frac{\partial(\rho v)}{\partial t} + \bar{\nabla} \cdot \left(\rho v \bar{V}\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y ....(2.56b)$$

$$\frac{\partial(\rho w)}{\partial t} + \bar{\nabla} \cdot \left(\rho w \bar{V}\right) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z ....(2.56c)$$

Governing equation summary

Energy equation

non-conservation form

$$\rho \frac{D}{Dt}\left(e+\frac{V^2}{2}\right)=\rho \dot{q}+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

$$-\frac{\partial(up)}{\partial x}-\frac{\partial(vp)}{\partial y}-\frac{\partial(wp)}{\partial z}+\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}$$

$$\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v\tau_{yy})}{\partial y}+\frac{\partial(v\tau_{zy})}{\partial z}+\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w\tau_{zz})}{\partial z}+\rho\bar{f}.\bar{V}\text{.......(2.66)}$$

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Governing equation summary

Energy equation

conservation form

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right]+\bar{\nabla}\cdot\left[\rho\left(e+\frac{V^2}{2}\right)\bar{V}\right]=\rho\dot{q}+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)$$

$$+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)-\frac{\partial(up)}{\partial x}-\frac{\partial(vp)}{\partial y}-\frac{\partial(wp)}{\partial z}$$

$$+\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}+\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v\tau_{yy})}{\partial y}$$

$$+\frac{\partial(v\tau_{zy})}{\partial z}+\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w\tau_{zz})}{\partial z}+\rho\bar{f}\cdot\bar{V}\text{.......(2.81)}$$

Physical Boundary Conditions

The above equations are very general. For example, they represent flow over an aircraft or flow in a hydraulic pump. To solve a specific problem much more information would be necessary. Some of them are listed below:

1. Boundary conditions (far field, solid boundary, etc)
2. Initial conditions (for unsteady problems)
3. Fluid medium (gas, liquid, non-Newtonian fluid, etc.)

BC specification depends on the type of flow we are interested in.

e. g., velocity boundary condition at the surface

"No slip condition" for viscous flow. All velocity components at the surface are zero.

Zero normal velocity of inviscid flow.

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Temperature BC at the wall.

Temperature, T_w, heat flux, $\dot{q}_w$, etc. can be specified.

Note

$$\dot{q}_w = -k \frac{\partial T}{\partial n}$$

If $\dot{q}_w$ is a known quantity, an expression for wall temperature, T_w can be written in terms of known quantities.

In this case the wall temperature will be obtained as part of the solution.

Conservation form of the equations
All equations can be expressed in the same generic form
fluxes can be written as

mass:	$\rho\bar{V}$
x-momentum:	$\rho u\bar{V}$
y-momentum:	$\rho v\bar{V}$
z-momentum:	$\rho w\bar{V}$
energy:	$\rho e\bar{V}$
total energy:	$\rho(e+\frac{V^2}{2})\bar{V}$

Conservation form contains divergence of these fluxes.
The set of equations can be written in the following vector form

$$\frac{\partial\bar{U}}{\partial t}+\frac{\partial\bar{F}}{\partial x}+\frac{\partial\bar{G}}{\partial y}+\frac{\partial\bar{H}}{\partial z}=\bar{J}..................(2.93)$$

$$\bar{U}=\left\{\begin{array}{l}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho\left(e+\frac{V^2}{2}\right)\end{array}\right\}............(2.94)$$

$$\bar{F} = \begin{Bmatrix} \rho u \\ \rho u^2 + p - \tau_{xx} \\ \rho vu - \tau_{xy} \\ \rho wu - \tau_{xz} \\ \rho\left(e + \frac{V^2}{2}\right)u + pu - k\frac{\partial T}{\partial x} - u\tau_{xx} - v\tau_{xy} - w\tau_{xz} \end{Bmatrix} \quad ....(2.95)$$

$$\bar{G} = \begin{Bmatrix} \rho v \\ \rho uv - \tau_{yx} \\ \rho v^2 + p - \tau_{yy} \\ \rho wv - \tau_{yz} \\ \rho\left(e + \frac{V^2}{2}\right)w + pv - k\frac{\partial T}{\partial y} - u\tau_{yx} - v\tau_{yy} - w\tau_{yz} \end{Bmatrix} \quad ....(2.96)$$

$$\bar{H} = \begin{Bmatrix} \rho w \\ \rho uw - \tau_{zx} \\ \rho vw - \tau_{zy} \\ \rho w^2 + p - \tau_{zz} \\ \rho\left(e + \frac{V^2}{2}\right)w + pw - k\frac{\partial T}{\partial z} - u\tau_{zx} - v\tau_{zy} - w\tau_{zz} \end{Bmatrix} \quad ....(2.97)$$

$$\bar{J} = \begin{Bmatrix} 0 \\ \rho f_x \\ \rho f_y \\ \rho f_z \\ \rho\left(uf_x + vf_y + wf_z\right) + \rho \dot{q} \end{Bmatrix} \text{..............(2.98)}$$

In the above U is called the solution vector
F, G, H are called flux vectors
J is called the source term vector

The problem is thus formulated as an unsteady problem.
Steady state solutions can be obtained asymptotically.
Once the flux variables are known from the solution, the “primitive” variables, u, v, w, p, e, etc. can be obtained from the flux variables.

Exercise write the vector form of the equations for inviscid flow (Euler equations).

Note that the following equations can be used to determine T

$$e = e(p, \rho)......................(2.112a)$$

$$e = c_v T = \frac{RT}{\gamma - 1} = \frac{R}{\gamma - 1}\frac{p}{\rho R}..........(2.112b)$$

$$e = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

Exercise: Write the corresponding equations for inviscid flow.

Observations:

1. Equations are coupled and non-linear
2. Conservation form contains divergence of some quantity on the LHS. This form is sometimes known as divergence form.
3. Normal and shear stress terms are functions of velocity gradient.
4. We have six unknowns and five equations (1 continuity + 3 momentum + 1 energy).
   For incompressible flow ρ can be treated as a constant.
   For compressible flow, the equation of state can be used as an additional equation for the solution.
5. The set of equations with viscosity included, is known as the Navier-Stokes equations.
6. The set of inviscid flow equations is also known as the Euler equations. These naming conventions are not strictly followed by everyone.

Program Completed

University of Missouri-Rolla