AE/ME 339 Computational Fluid Dynamics (CFD)

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<u>The pressure correction formula (6.8.4)</u>

Calculation of p'.
Conservation form of the momentum equations are as follows:
Note that these equations can be obtained from the non-conservation form by using the continuity equation.
See Patankar for a slightly different formulation.

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)........................(6.88)$$

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho v^2)}{\partial y}+\frac{\partial(\rho uv)}{\partial x}=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)........................(6.89)$$

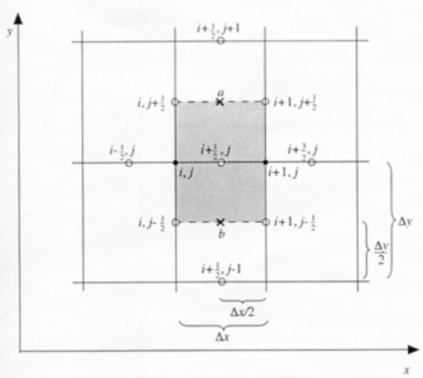

FIG. 6.16
Computational module for the *x*-momentum equation. The filled-in area is an effective cont

Figure 6.16 shows a staggered grid where the pressures are calculated at the solid grid points and the velocities are evaluated at the open grid points.
Write the difference equation for Eq. 6.88 around the point (i+1/2 , j) shown in the figure.
We need average values of v at points a and b. It is obtained by interpolation.

at Point a:

$$\bar{v}_{j+1/2} \equiv \frac{1}{2}\left(v_{i,j+1/2} + v_{i+1,j+1/2}\right)..................(6.90a)$$

at Point b:

$$\bar{v}_{j-1/2} \equiv \frac{1}{2}\left(v_{i,j-1/2} + v_{i+1,j-1/2}\right)..................(6.90b)$$

Difference representation for Eq. 6.88 centered around point (i+1/2, j)

$$\frac{(\rho u)^{n+1}_{i+1/2,j} - (\rho u)^{n}_{i+1/2,j}}{\Delta t} = -\left[\frac{(\rho u^2)^n_{i+3/2,j} - (\rho u^2)^n_{i-1/2,j}}{2\Delta x} + \frac{(\rho u \bar{v})^n_{i+1/2,j+1} - (\rho u v)^n_{i+1/2,j-1}}{2\Delta y}\right] - \frac{p^n_{i+1,j} - p^n_{i,j}}{\Delta x}$$

$$+\mu\left[\frac{u^n_{i+3/2,j} - 2u^n_{i+1/2,j} + u^n_{i-1/2,j}}{(\Delta x)^2} + \frac{u^n_{i+1/2,j+1} - 2u^n_{i+1/2,j} + u^n_{i+1/2,j-1}}{(\Delta y)^2}\right] \text{......(6.91)}$$

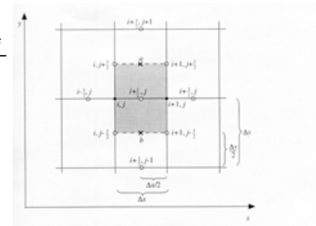

FIG. 6.16
Computational module for the x-momentum equation. The filled-in area is an effective cont

or

$$(\rho u)^{n+1}_{i+1/2,j} = (\rho u)^{n}_{i+1/2,j} + A\Delta t - \frac{\Delta t}{\Delta x}\left(p^n_{i+1,j} - p^n_{i,j}\right) \text{..........(6.92)}$$

where

$$A = -\left[\frac{(\rho u^2)^n_{i+3/2,j} - (\rho u^2)^n_{i-1/2,j}}{2\Delta x} + \frac{(\rho u \bar{v})^n_{i+1/2,j+1} - (\rho u v)^n_{i+1/2,j-1}}{2\Delta y}\right]$$

$$+\mu\left[\frac{u^n_{i+3/2,j} - 2u^n_{i+1/2,j} + u^n_{i-1/2,j}}{(\Delta x)^2} + \frac{u^n_{i+1/2,j+1} - 2u^n_{i+1/2,j} + u^n_{i+1/2,j-1}}{(\Delta y)^2}\right]$$

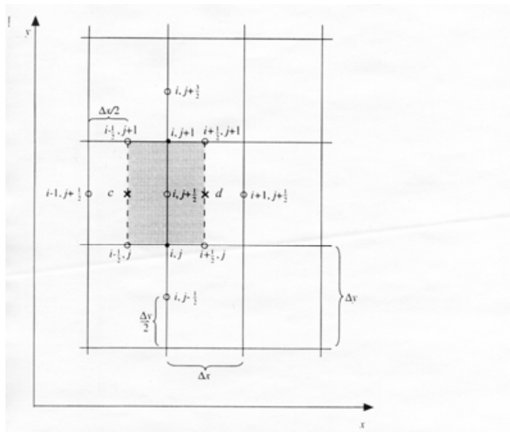

FIG. 6.17
Computational module for the *y*-momentum equation. The filled-in area is an effective control volume.

At Point c: $$u = \frac{1}{2}\left(u_{i-1/2,\,j} + u_{i-1/2,\,j+1}\right)$$

At Point d: $$\bar{u} = \frac{1}{2}\left(u_{i+1/2,\,j} + u_{i+1/2,\,j+1}\right)$$

Use forward difference in time and central difference in space in Eq. 6.89 gives

$$\left(\rho v\right)_{i,\,j+1/2}^{n+1} = \left(\rho v\right)_{i,\,j+1/2}^{n} + B\Delta t - \frac{\Delta t}{\Delta x}\left(p_{i,\,j+1}^{n} - p_{i,\,j}^{n}\right) \text{..........(6.93)}$$

where

$$B = -\left[\frac{\left(\rho v \bar{u}\right)^n_{i+1,j+1/2} - \left(\rho u v\right)^n_{i-1,j+1/2}}{2\Delta x} + \frac{\left(\rho v^2\right)^n_{i,j+3/2} - \left(\rho v^2\right)^n_{i,j-1/2}}{2\Delta y} \right]$$

$$+\mu \left[\frac{v^n_{i+1,j+1/2} - 2v^n_{i,j+1/2} + v^n_{i-1,j+1/2}}{\left(\Delta x\right)^2} + \frac{v^n_{i,j+3/2} - 2v^n_{i,j+1/2} + v^n_{i,j-1/2}}{\left(\Delta x\right)^2} \right]$$

The staggered grid approach used can be seen from Figures 6.16 and 6.17. The shaded areas in these two figures represent finite volumes that would be used in the formulation by Patankar and Spalding. Note that the two areas don't completely overlap.

At the beginning of each iteration p = p*. Therefore, for this step Eqs. 6.92 and 6.93 become

$$\left(\rho u^*\right)^{n+1}_{i+1/2,j} = \left(\rho u^*\right)^n_{i+1/2,j} + A^*\Delta t - \frac{\Delta t}{\Delta x}\left(p^*_{i+1,j} - p^*_{i,j}\right) \text{..........(6.94)}$$

$$\left(\rho v^*\right)^{n+1}_{i,j+1/2} = \left(\rho v^*\right)^n_{i,j+1/2} + B^*\Delta t - \frac{\Delta t}{\Delta y}\left(p^*_{i,j+1} - p^*_{i,j}\right) \text{..........(6.95)}$$

Subtract Eq. 6.94 from Eq. 6.92 and Eq. 6.95 from Eq. 6.93 to get the following:

$$\left(\rho u^{'}\right)^{n+1}_{i+1/2,j} = \left(\rho u^{'}\right)^{n}_{i+1/2,j} + A^{'}\Delta t - \frac{\Delta t}{\Delta x}\left(p^{'}_{i+1,j} - p^{'}_{i,j}\right) \text{..........(6.96)}$$

where

$$\left(\rho u^{'}\right)^{n+1}_{i+1/2,j} = \left(\rho u\right)^{n+1}_{i+1/2,j} - \left(\rho u^{*}\right)^{n+1}_{i+1/2,j}$$

$$\left(\rho u^{'}\right)^{n}_{i+1/2,j} = \left(\rho u\right)^{n}_{i+1/2,j} - \left(\rho u^{*}\right)^{n}_{i+1/2,j}$$

$$A^{'} = A - A^{*}, \qquad p^{'}_{i+1,j} = p_{i+1,j} - p^{*}_{i+1,j}, \qquad p^{'}_{i,j} = p_{i,j} - p^{*}_{i,j}$$

$$\left(\rho v^{'}\right)^{n+1}_{i,j+1/2} = \left(\rho v^{'}\right)^{n}_{i,j+1/2} + B^{'}\Delta t - \frac{\Delta t}{\Delta y}\left(p^{'}_{i,j+1} - p^{'}_{i,j}\right) \text{..........(6.97)}$$

where

$$\left(\rho v^{'}\right)^{n+1}_{i,j+1/2} = \left(\rho v\right)^{n+1}_{i,j+1/2} - \left(\rho v^{*}\right)^{n+1}_{i,j+1/2}$$

$$\left(\rho v^{'}\right)^{n}_{i,j+1/2} = \left(\rho v\right)^{n}_{i,j+1/2} - \left(\rho v^{*}\right)^{n}_{i,j+1/2}$$

$$B^{'} = B - B^{*}, \qquad p^{'}_{i,j+1} = p_{i,j+1} - p^{*}_{i,j+1}, \qquad p^{'}_{i,j} = p_{i,j} - p^{*}_{i,j}$$

Eqs. 6.96 and 6.97 are the x and y momentum equations written in terms of the velocity and pressure correction terms.
The next important step is to get an expression for pressure correction by using the condition that the velocity field must satisfy the conservation of mass equation.

The p’ formula that will be used is not an exact representation. It is devised such that when convergence is achieved:

$$p' \to 0$$

and the formula for p’ tends to the physically correct continuity equation.

Patankar sets $A', B', (\rho u')^n, (\rho v')^n$ to zero in Eqs. 6.96 and 6.97 which would yield the following equations.

$$\left(\rho u'\right)_{i+1/2,j}^{n+1} = -\frac{\Delta t}{\Delta x}\left(p'_{i+1,j} - p'_{i,j}\right)^n \text{..............(6.98)}$$

$$\left(\rho v'\right)_{i,j+1/2}^{n+1} = -\frac{\Delta t}{\Delta y}\left(p'_{i,j+1} - p'_{i,j}\right)^n \text{............(6.99)}$$

Recall that

$$\left(\rho u'\right)_{i+1/2,j}^{n} = \left(\rho u\right)_{i+1/2,j}^{n} - \left(\rho u^*\right)_{i+1/2,j}^{n}$$

Eq. 6.98 can be written as

$$(\rho u)^{n+1}_{i+1/2,j} = (\rho u^*)^{n+1}_{i+1/2,j} - \frac{\Delta t}{\Delta x}\left(p'_{i+1,j} - p'_{i,j}\right)^n \text{..................(6.100)}$$

Similarly, Eq. 6.99 becomes

$$(\rho v)^{n+1}_{i,j+1/2} = (\rho v^*)^{n+1}_{i,j+1/2} - \frac{\Delta t}{\Delta y}\left(p'_{i,j+1} - p'_{i,j}\right)^n \text{.............(6.101)}$$

The continuity equation centered around the point (i,j) using central differencing (CD) becomes

$$\frac{(\rho u)_{i+1/2,j} - (\rho u)_{i-1/2,j}}{\Delta x} + \frac{(\rho v)_{i,j+1/2} - (\rho v)_{i,j-1/2}}{\Delta y} = 0\text{....................(6.102)}$$

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Substitute Eqs. (6.100) and and (6.101) in Eq. (6.102) and dropping the superscript gives

$$\frac{(\rho u^*)_{i+1/2,j} - \Delta t/\Delta x\left(p'_{i+1,j} - p'_{i,j}\right) - (\rho u^*)_{i-1/2,j} + \Delta t/\Delta x\left(p'_{i,j} - p'_{i-1,j}\right)}{\Delta x}$$

$$\frac{(\rho v^*)_{i,j+1/2} - \Delta t/\Delta y\left(p'_{i,j+1} - p'_{i,j}\right) - (\rho v^*)_{i,j-1/2} + \Delta t/\Delta y\left(p'_{i,j} - p'_{i,j-1}\right)}{\Delta y} = 0\text{....(6.103)}$$

Eq. (6.103) can be rearranged to give (see next slide for expressions for a, b, c and d)

$$ap'_{i,j} + bp'_{i+1,j} + bp'_{i-1,j} + cp'_{i,j+1} + cp'_{i,j-1} + d = 0\text{............(6.104)}$$

Where a, b, c and d are given by the following expressions

$$a = 2\left[\frac{\Delta t}{(\Delta x)^2} + \frac{\Delta t}{(\Delta y)^2}\right] \qquad b = -\frac{\Delta t}{(\Delta x)^2} \qquad c = -\frac{\Delta t}{(\Delta y)^2}$$

$$d = \frac{1}{\Delta x}\left[\left(\rho u^*\right)_{i+1/2,j} - \left(\rho u^*\right)_{i-1/2,j}\right] + \frac{1}{\Delta y}\left[\left(\rho v^*\right)_{i,j+1/2} - \left(\rho v^*\right)_{i,j-1/2}\right]$$

Eq. (6.104) gives the pressure correction.

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The SIMPLE algorithm:
The pressure correction formula (Eq. 6.104) is approximate because we set $A', B', (\rho u')^n, (\rho v')^n$ equal to zero. Hence the term "Semi-implicit" in the name.
This makes the pressure effect to be localized.

for the staggered grid given in Figure 6.15

1. Guess $(p^*)^n$ at all pressure nodes and set $(\rho u^*)^n, (\rho v^*)^n$ arbitrarily at the appropriate velocity nodes.
2. Solve for $(\rho u^*)^{n+1}, (\rho v^*)^{n+1}$ using Eqs. (6,94) and (6.95) respectively.
3. Substitute these values of $(\rho u^*)^{n+1}, (\rho v^*)^{n+1}$ in Eq. (104) and solve for p' at the interior nodes (boundary nodes will be treated separately). Relaxation procedure would work.
4. The values of (p)n+1obtained in the previous step are used for
5. Calculate $p^{n+1} = (p^*)^n + p'$ at all nodes.
   solving the momentum equations.
6. Repeat steps 2-5 until convergence criteria are satisfied.

The superscript (n) and (n+1) used in the above equations are pseudo-time in the sense that solution obtained from this procedure will not be "**time-accurate**."
Therefore, the method essentially is a "**time-dependent**" method for steady state problems.
(n) and (n+1) therefore, can be thought of as representing sequential iteration steps.

The above procedure can cause the solution to diverge. Extensive use of "under-relaxation factors" is employed as a remedy. The following equation is an example of how under-relaxation factor can be used.

$$p^{n+1} = p^n + \alpha_p p' ....(6.106)$$

where a is the under relaxation factor.

$$0 \le \alpha_p \le 1$$

Boundary conditions for pressure correction method (6.8.6)

Insert Figure 6.18

At the inflow boundary:
p and v are specified, u is allowed to float. Therefore, p' = 0 at the inflow boundary.

Outflow boundary:
p is specified and u and v are allowed to float.

At the walls:
No slip condition gives all velocities to be zero.
The y-momentum (Eq. 6.79) equation at the wall can be written as:

$$\left(\frac{\partial p}{\partial y}\right)_w = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)_w \text{......................(6.107)}$$

Since v = 0 at the wall, the first term on the RHS will be zero. Also we approximate the second term to be zero because it is usually small.

Therefore, we have the following approximate condition at the wall

$$\left(\frac{\partial p}{\partial y}\right)_w = 0 \text{................} (6.108)$$

Program Completed

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