



AE/ME 339 Computational Fluid Dynamics (CFD)

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9 November 2004

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Computational Fluid Dynamics (AE/ME 339)

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Flow over a cylinder is a classical problem

Potential flow over a cylinder is equivalent to a doublet in uniform flow

See potential flow theory for details.

The analytical solution for ϕ is as follows

$$\phi = \underbrace{U_{\infty} r \cos(\theta)}_{\text{uniform flow}} + \underbrace{U_{\infty} \frac{R^2}{r} \cos(\theta)}_{\text{doublet flow}} = U_{\infty} r \cos(\theta) \left(1 + \frac{R^2}{r^2} \right)$$

Velocity component at any point is given by

$$\vec{V} = \hat{e}_r u_r + \hat{e}_{\theta} u_{\theta} = \nabla \phi \equiv \hat{e}_r \frac{\partial \phi}{\partial r} + \hat{e}_{\theta} \frac{1}{r} \frac{\partial \phi}{\partial \theta}$$

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$u_r =$

Differentiation yields

$$u_r = U_\infty \cos(\theta) \left(1 - \frac{R^2}{r^2} \right)$$

$$u_\theta = -U_\infty \sin(\theta) \left(1 + \frac{R^2}{r^2} \right)$$

Pressure distribution can be represented in terms of the pressure Coefficient C_p

$$C_p = \frac{p - p_\infty}{\frac{1}{2} \rho U_\infty^2}$$

Bernoulli's equation

$$p + (1/2)\rho V^2 = p_\infty + (1/2)\rho U_\infty^2$$

$$\therefore C_p = 1 - \left(\frac{V}{U_\infty} \right)^2$$

Recall

$$V^2 = u_r^2 + u_\theta^2$$

Surface pressure distribution can now be obtained by substituting $r=R$

$$u_r|_{r=R} = 0$$

$$u_\theta|_{r=R} = -2U_\infty \sin(\theta)$$

$$C_p|_{r=R} = 1 - 4\sin^2(\theta)$$

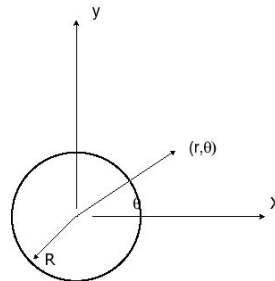
Note that the surface pressure distribution is independent of the cylinder radius.

Flow Over a Cylinder

The transformation from (x,y) to polar coordinates (r, θ) will give boundary-fitted coordinates. (Recall $r = R$ (constant) is the cylinder surface.

Here the domain is between $r = R$ and $r = \text{infinity}$.

Computationally, the outer boundary needs to be finite, say $r = r_1$, where the far field (free-stream) conditions can be applied.



We can also use the transformation $\sigma = 1/r$. With this transformation, $\sigma = 1/R$ represents the cylinder surface and $\sigma = 0$ represents conditions at infinity.

In the new (σ, θ) system, the computational space lies between $\sigma = 0$ (far field) and $1/R$ (surface).

First consider the polar coordinates. The relationship between the two systems is as follows:

$$\begin{aligned}x &= r \cos(\theta) \\y &= r \sin(\theta)\end{aligned}$$

Transformation metrics:

These transformations can be checked by applying them to Laplace's equation, which in polar coordinates is given by

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0 \dots (1)$$

$$\frac{\partial x}{\partial r} = \cos(\theta), \quad \frac{\partial^2 x}{\partial r^2} = 0$$

$$\frac{\partial x}{\partial \theta} = -r \sin(\theta), \quad \frac{\partial^2 x}{\partial \theta^2} = -r \cos(\theta)$$

$$\frac{\partial y}{\partial r} = \sin(\theta), \quad \frac{\partial^2 y}{\partial r^2} = 0$$

$$\frac{\partial y}{\partial \theta} = r \cos(\theta), \quad \frac{\partial^2 y}{\partial \theta^2} = -r \sin(\theta)$$

General form of the transformation

$$\frac{\partial^2}{\partial r^2} = \left(\frac{\partial^2 x}{\partial r^2} \right) \frac{\partial}{\partial x} + \left(\frac{\partial^2 y}{\partial r^2} \right) \frac{\partial}{\partial y} + \left(\frac{\partial x}{\partial r} \right)^2 \frac{\partial^2}{\partial x^2} +$$

$$\left(\frac{\partial y}{\partial r} \right)^2 \frac{\partial^2}{\partial y^2} + 2 \left(\frac{\partial x}{\partial r} \right) \left(\frac{\partial y}{\partial r} \right) \frac{\partial^2}{\partial x \partial y}$$

Substituting for the metrics give the following

Substituting for the metrics give the following

$$\frac{\partial^2}{\partial r^2} = \underbrace{\left(\frac{\partial^2 x}{\partial r^2} \right)}_0 \frac{\partial}{\partial x} + \underbrace{\left(\frac{\partial^2 y}{\partial r^2} \right)}_0 \frac{\partial}{\partial y} + \underbrace{\left(\frac{\partial x}{\partial r} \right)^2}_{\cos^2(\theta)} \frac{\partial^2}{\partial x^2} +$$

$$\underbrace{\left(\frac{\partial y}{\partial r} \right)^2}_{\sin^2(\theta)} \frac{\partial^2}{\partial y^2} + 2 \underbrace{\left(\frac{\partial x}{\partial r} \right)}_{\cos(\theta)} \underbrace{\left(\frac{\partial y}{\partial r} \right)}_{\sin(\theta)} \frac{\partial^2}{\partial x \partial y}$$

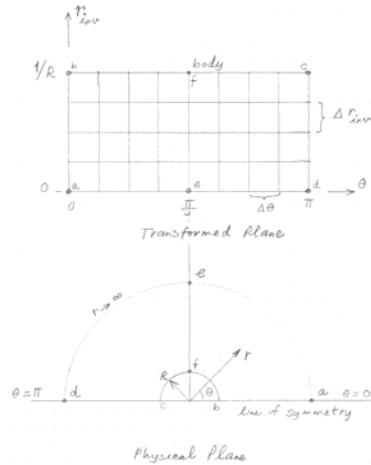
Similarly, get expressions for the other terms

$$\frac{\partial^2}{\partial \theta^2}, \frac{\partial}{\partial r}$$

And substitute in Laplace's equation in polar coordinates to transform
The equations to cartesian coordinates.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \dots\dots (2)$$

For this problem the transformation $\sigma = 1/r$ can be used to transform
the original equation in terms of σ .
Also since the flow is symmetric about $\theta = 0$ line, we need to solve
only one half of the flow domain.



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The governing equation can now be written in terms of σ as follows

$$\frac{\partial}{\partial r} = \frac{\partial \sigma}{\partial r} \frac{\partial}{\partial \sigma} = -\sigma^2 \frac{\partial}{\partial \sigma}$$

$$\frac{\partial^2}{\partial r^2} = \frac{\partial}{\partial r} \left(\frac{\partial}{\partial r} \right) = \frac{\partial}{\partial r} \left(-\sigma^2 \frac{\partial}{\partial \sigma} \right) =$$

$$-\sigma^2 \frac{\partial}{\partial \sigma} \left(-\sigma^2 \frac{\partial}{\partial \sigma} \right) = \sigma^4 \frac{\partial^2}{\partial \sigma^2} + 2\sigma^3 \frac{\partial}{\partial \sigma}$$

Using the above relations in Laplace's equation gives

$$\sigma^2 \frac{\partial^2 \phi}{\partial \sigma^2} + \sigma \frac{\partial \phi}{\partial \sigma} + \frac{\partial^2 \phi}{\partial \theta^2} = 0$$

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The above equation needs to be solved on the rectangular grid shown in the figure.

Before starting the solution, the boundary conditions also must be transformed.

At $r = R$:

$r = R$:

$$\frac{\partial \phi}{\partial r} = u_r = 0$$

$$\frac{\partial \phi}{\partial r} \Big|_{r=R} = -\sigma^2 \frac{\partial \phi}{\partial \sigma} \Big|_{\sigma=(1/R)} = 0$$

$$\therefore \frac{\partial \phi}{\partial \sigma} \Big|_{\sigma=(1/R)} = 0$$

$r \rightarrow \infty$:

$$\phi = U_\infty r \cos(\theta)$$

$$r \rightarrow \infty: \frac{\partial \phi}{\partial r} = -\sigma^2 \frac{\partial \phi}{\partial \sigma} = U_\infty \cos(\theta)$$

$$\text{or: } \phi = \frac{U_\infty}{\sigma} \cos(\theta)$$

Along the line of symmetry $U_\theta = 0$.

$$\frac{1}{r} \frac{\partial \phi}{\partial \theta} = 0$$

$$\text{i.e., } \frac{\partial \phi}{\partial \theta} = 0$$



Program Completed

University of Missouri-Rolla

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