AE/ME 339
Computational Fluid Dynamics (CFD)

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Burger's Equation

(Ref. Tannehill, Anderson and Pletcher, *Computational Fluid Mechanics and Heat Transfer*, Sect, 4.4, 1997)

It is a model equation used to test finite difference techniques

Inviscid and viscous forms can be used

Has a time dependent term, non-linear term similar to the convection term, and a viscous dissipation term

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{time-dependent term}} + \underbrace{u\frac{\partial u}{\partial x}}_{\text{convection term}} = \underbrace{\nu\frac{\partial^2 u}{\partial x^2}}_{\text{viscous diffusion term}} \quad ......(1)$$

Equation (1) is parabolic when the viscous dissipation term is included. When the RHS term = 0, the equation is hyperbolic, which gives the following inviscid form.

Inviscid form

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{time-dependent term}} + \underbrace{u\frac{\partial u}{\partial x}}_{\text{convection term}} = 0 \text{..............(2)}$$

Equation 2 can be thought of as the non-linear wave equation, where each point on the wave can propagate with a different speed leading to the formation of shock waves.
Shock formation is a non-linear phenomenon.





Linear wave equation

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{time-dependent term}} + \underbrace{a\frac{\partial u}{\partial x}}_{\text{convection term}} = 0 \text{..............(3)}$$

Governs propagation of acoustic waves (linearized shock waves) where a is the constant wave propagation speed (speed of sound)

Traveling Discontinuity (shock propagation)
Problem for Burger's Equation.
Consider the initial data shown below

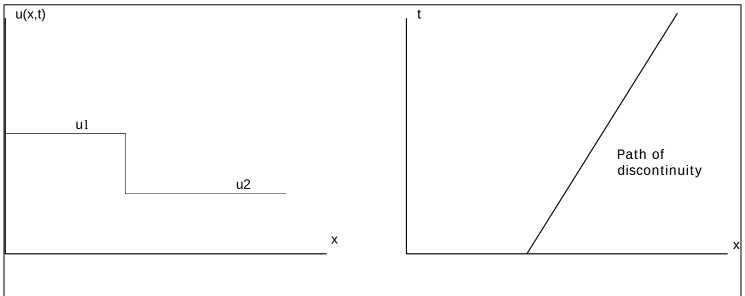



Figure 1. Traveling discontinuity proble for Burger's equation





It can be shown that the discontinuity travels with the speed (Tannehill et al., 1997)

$$u = \frac{dx}{dt} = \frac{u_1 + u_2}{2} \qquad (6)$$

See Figure 1 (previous slide).

Consider a different initial data u(x,0) shown in Figure 2.

The solution will show centered expansion

The characteristic equation is given by

$$\frac{dt}{dx} = \frac{1}{u} \qquad (7)$$

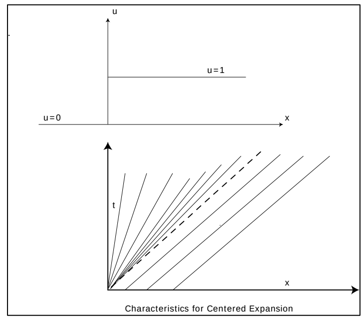



Figure 1.





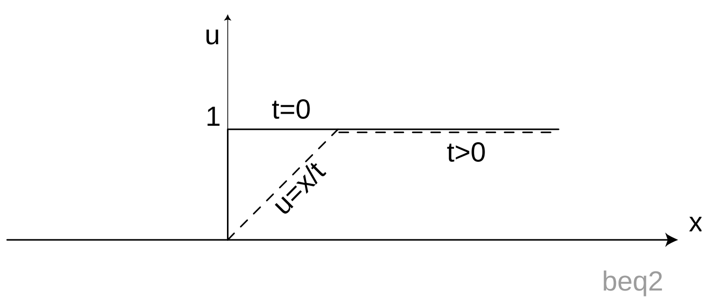
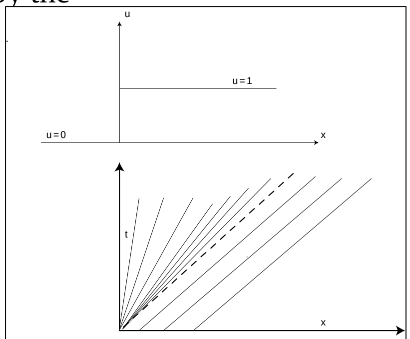



Figure 2. Solution at t = 0 and t > 0

Figure 1 shows the characteristic diagram plotted in the (x, t) space.
Bounded by the x = 0 (vertical) line and the characteristic denoted by the dashed line.




Characteristics for Centered Expansion





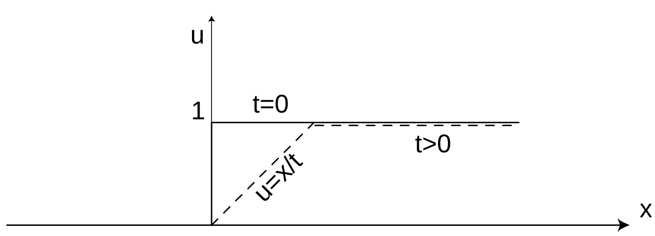
Solution can be written as

$$u = 0 \qquad x \le 0$$

$$u = \frac{x}{t} \qquad 0 < x < u_0 t \quad \text{(recall: } \frac{dt}{dx} = \frac{1}{u_0}\text{)}$$

$$u = u_0 \qquad x \ge u_0 t$$

Graph showing solution at t = 0 and t > 0




beq2

The initial distribution of u results in a centered expansion where the width of the expansion grows linearly with time.

The above solutions can now be used to evaluate finite difference algorithms.





## Beam and Warming method
## Outline

1. Write Taylor series for $u_j^{n+1}$ about $u_j^n$ and $u_j^n$ about $u_j^{n+1}$
2. Subtract the second from the first
3. Write Taylor Series for $\left(u_{tt}\right)_j^{n+1}$ and substitute in Eq. (10)
4. Replace $u_t$ with $-au_x$
5. Use central difference for $u_x$ on the RHS
6. Drop third order terms

## Beam-Warming Method (Beam and Warming, 1976)

Let us consider the following equation

$$\frac{\partial u}{\partial t}+\frac{\partial F(u)}{\partial x}=0 \qquad (3)$$

where $F = F(u)$

Rewrite as $\frac{\partial u}{\partial t}+A\frac{\partial u}{\partial x}=0$ (4)

where $A = \frac{\partial F}{\partial u}$ (5)

Equation (3) could represent a vector

in which case $A = \frac{\partial F_i}{\partial u_j}$ is the Jacobian matrix





Consider the the following two Taylor series expansions

$$u_j^{n+1}=u_j^n+\Delta t(u_t)_j^n+\frac{(\Delta t)^2}{2}(u_{tt})_j^n+\frac{(\Delta t)^3}{6}(u_{ttt})_j^n+\ldots \qquad (8)$$

$$u_j^n=u_j^{n+1}-\Delta t(u_t)_j^{n+1}+\frac{(\Delta t)^2}{2}(u_{tt})_j^{n+1}-\frac{(\Delta t)^3}{6}(u_{ttt})_j^{n+1}+\ldots \qquad (9)$$

Subtract Equation (9) from Equation (8)

$$u_j^{n+1}-u_j^n=u_j^n-u_j^{n+1}+\Delta t\left\{(u_t)_j^n+(u_t)_j^{n+1}\right\}+$$

$$\frac{(\Delta t)^2}{2}\left\{(u_{tt})_j^n-(u_{tt})_j^{n+1}\right\}+O[(\Delta t)^3] \qquad (10)$$

$(u_{tt})_j^{n+1}$ can be substituted in Eq. (10) using the following Taylor series expansion

$$(u_{tt})_j^{n+1} = (u_{tt})_j^n + \Delta t(u_{ttt})_j^n + ... \qquad (11)$$

Eq. (10) becomes

$$2u_j^{n+1} = 2u_j^n + \Delta t\left\{(u_t)_j^n + (u_t)_j^{n+1}\right\} + \frac{(\Delta t)^2}{2}\left\{(u_{tt})_j^n - (u_{tt})_j^n - \Delta t(u_{ttt})_j^n\right\} + O[(\Delta t)^3] \qquad (12)$$

Which reduces to

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{2}\left\{(u_t)_j^n + (u_t)_j^{n+1}\right\} + O[(\Delta t)^3] \qquad (13)$$





Now we substitute the wave equation $u_t = -au_x$ to get the following

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2}\left\{(u_x)_j^n + (u_x)_j^{n+1}\right\} + O[(\Delta t)^3] \qquad (14)$$

and now replace the $u_x$ terms by 2nd order central differences

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{4}\left\{(u_x)_{j+1}^n - (u_x)_{j-1}^n + (u_x)_{j+1}^{n+1} - (u_x)_{j-1}^{n+1}\right\} + O[(\Delta t)^3] \qquad (15)$$

The method is 2nd order accurate ($\varepsilon = O[(\Delta t)^2,(\Delta x)^2]$) and unconditionally stable for all time steps.
A tridiagonal system must be solved for each time step.





## Summary

1. Write Taylor series for $u_j^{n+1}$ about $u_j^n$ and $u_j^n$ about $u_j^{n+1}$
2. Subtract the second from the first
3. Write Taylor Series for $\left(u_{tt}\right)_j^{n+1}$ and substitute in Eq. (10)
4. Replace $u_t$ with $-au_x$
5. Use central difference for $u_x$ on the RHS
6. Drop third order terms

The Beam-Warming method can now be applied to the inviscid Burger's equation





Substituting in Eq. (14) using Eq. (3) gives

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2}\left\{\left(\frac{\partial F}{\partial x}\right)^n + \left(\frac{\partial F}{\partial x}\right)^{n+1}\right\} \quad (15)$$

The above is a non-linear problem since $F = F(u)$.

Linearization or iteration is therefore necessary

Beam and Warming (1976) suggested the following

$$F^{n+1} \approx F^n + \left(\frac{\partial F}{\partial u}\right)^n (u^{n+1} - u^n) = F^n + A^n(u^{n+1} - u^n) \quad (16)$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2}\left\{2\left(\frac{\partial F}{\partial x}\right)^n + \frac{\partial}{\partial x}\left[A^n(u^{n+1} - u^n)\right]\right\} \quad (17)$$

Replacing the x-derivatives using 2nd order CD would yield the following

$$-\frac{1}{4}\frac{\Delta t}{\Delta x}A_{j-1}^{n}u_{j-1}^{n+1}+u_{j}^{n+1}+\frac{1}{4}\frac{\Delta t}{\Delta x}A_{j+1}^{n}u_{j+1}^{n+1}=$$

$$-\frac{\Delta t}{\Delta x}\frac{F_{j+1}^{n}-F_{j-1}^{n}}{2}-\frac{1}{4}\frac{\Delta t}{\Delta x}A_{j-1}^{n}u_{j-1}^{n}+u_{j}^{n}+\frac{1}{4}\frac{\Delta t}{\Delta x}A_{j+1}^{n}u_{j+1}^{n} \quad (18)$$

The Jacobian A has a single element for the Burger's equation.

Eq. (18) represents linear tridiagonal system.

Solution by Thomas algorithm is feasible.





Beam and Warming suggests the following explicit artificial viscosity term

$$D=-\frac{\omega}{8}\left(u_{j+2}^{n}+u_{j+1}^{n}+u_{j}^{n}+u_{j-1}^{n}+u_{j-2}^{n}\right) \quad (19)$$

Recommended values of $\omega$ lie in the range

$$0\le\omega\le1$$

## Delta Form

Sometimes it is better to write the equation for change in the variable from time level n to (n+1). Eq. (18) then becomes

$$-\frac{1}{4}\frac{\Delta t}{\Delta x}A_{j-1}^{n}\Delta u_{j-1}^{n+1}+\Delta u_{j}^{n+1}+\frac{1}{4}\frac{\Delta t}{\Delta x}A_{j+1}^{n}\Delta u_{j+1}^{n+1}=$$

$$-\frac{\Delta t}{\Delta x}\frac{F_{j+1}^{n}-F_{j-1}^{n}}{2} \qquad (20)$$





The delta form reduces the number of arithmatic operations since the RHS has only one term. Also round-off error will be smaller in this case.

# Some Examples





## Solution of Burger's equation

Use MacCormack's method to solve inviscid Burger's equation using a mesh with 51 points in the x-direction. Solve the equation for a right propagating discontinuity with u = 1 at the first 11 nodes and u = 0 at the rest of the nodes.
Use Courant number = 1.

Solution

MacCormack's method

$$\overline{u}_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1}^n - F_j^n\right)$$

$$u_j^{n+1} = \frac{1}{2}\left[u_j^n + \overline{u}_j^{n+1} - \frac{\Delta t}{\Delta x}\left(\overline{F}_j^n - \overline{F}_{j-1}^n\right)\right]$$

# *Program Completed*

## *University of Missouri-Rolla*