AE/ME 339 Computational Fluid Dynamics (CFD)

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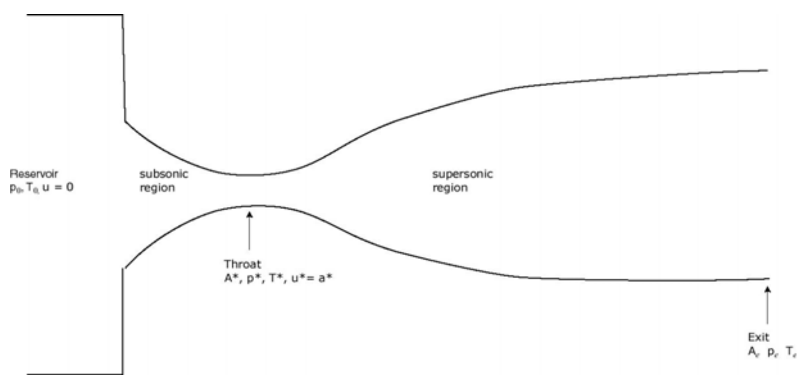
Quasi One-Dimensional Nozzle Flow



Continuity: $$\frac{\partial}{\partial t}(\rho A)+\frac{\partial}{\partial x}(\rho A u)=0$$

Momentum: $$\frac{\partial}{\partial t}(\rho u A)+\frac{\partial}{\partial x}(\rho u^2 A)=-A\frac{\partial p}{\partial x}$$

Energy: $$\frac{\partial}{\partial t}(\rho e A)+\frac{\partial}{\partial x}(\rho e u A)=-p\frac{\partial}{\partial x}(Au)$$

Equations in conservation form (Dimensional)

$$\frac{\partial}{\partial t}(\rho A)+\frac{\partial}{\partial x}(\rho A u)=0$$

$$\frac{\partial}{\partial t}(\rho A u)+\frac{\partial}{\partial x}(\rho A u^2+pA)=p\frac{\partial A}{\partial x}$$

$$\frac{\partial}{\partial t}\left(\rho A(e+\frac{u^2}{2})\right)+\frac{\partial}{\partial x}\left(\rho A(e+\frac{u^2}{2})u+pAu\right)=0$$

The above equations can be solved in the same manner as the shock tube flow equations.

Conservation Form

Non-dimensionalize the governing equations as follows:

$$\theta = T / T_0$$
$$\sigma = \rho / \rho_0$$
$$\xi = x / L$$
$$a_0 = \sqrt{\gamma R T_0}$$
$$\upsilon = V / a_0$$
$$\alpha = A / A^*$$

Governing Equations in Non-dimensional Variables

$$\frac{\partial(\sigma\alpha)}{\partial\tau} + \frac{\partial(\sigma\alpha\upsilon)}{\partial\xi} = 0$$

$$\frac{\partial(\sigma\alpha\upsilon)}{\partial\tau} + \frac{\partial\left(\sigma\alpha\upsilon^2 + \frac{1}{\gamma}\phi\alpha\right)}{\partial\xi} = \frac{1}{\gamma}\phi\frac{\partial\alpha}{\partial\xi}$$

$$\frac{\partial\left[\alpha\sigma\left(\frac{\theta}{\gamma-1} + \gamma\frac{\upsilon^2}{2}\right)\right]}{\partial\tau} + \frac{\partial\left[\alpha\sigma\phi\left(\frac{\theta}{\gamma-1} + \gamma\frac{\upsilon^2}{2}\right) + \alpha\phi\upsilon\right]}{\partial\xi} = 0$$

Vector Elements

$$U_1 = \alpha\sigma$$
$$U_2 = \alpha\sigma\upsilon$$
$$U_3 = \alpha\sigma\left(\frac{\theta}{\gamma - 1} + \frac{\gamma}{2}\upsilon^2\right)$$
$$F_1 = \alpha\sigma\upsilon$$
$$F_2 = \alpha\sigma\upsilon^2 + \frac{1}{\gamma}\alpha\phi, \qquad J_2 = \frac{1}{\gamma}\phi\frac{\partial\alpha}{\partial\xi}$$
$$F_3 = \alpha\sigma\upsilon\left(\frac{\theta}{\gamma - 1} + \frac{\gamma}{2}\upsilon^2\right) + \alpha\phi\upsilon$$

Relationship between U vector and F vector

$$F_1 = U_2$$
$$F2 = \frac{U_2^2}{U_1} + \frac{\gamma - 1}{\gamma}\left(U_3 - \frac{\gamma}{2}\frac{U_2^2}{U_1}\right), \qquad \mathrm{J}_2 = \frac{\gamma - 1}{\gamma}\left(U_3 - \frac{\gamma}{2}\frac{U_2^2}{U_1}\right)\frac{\partial(\ln\alpha)}{\partial\xi}$$
$$F_3 = \gamma\frac{U_2 U_3}{U_1} - \frac{\gamma(\gamma - 1)}{2}\frac{U_2^3}{U_1^2}$$

Courant Number $C = (u + a)\dfrac{\Delta t}{\Delta x}$, can be used to determine the time step that satisfies the stability criterion. $C \le 1$ ensures that the solution will be stable. Note the above stability criterion is an empirical relation and therefore, it would be safe to use a time step for which the value of C will be well below unity.

Boundary conditions

The boundary conditions can be established using the theory of the method of characteristics (Tannehill, et al. 1997).

At the subsonic inflow boundary we set the values of T and ρ and use linear extrapolation for u.

At the nozzle exit, T, ρ, and u are obtained by using linear extrapolation.

An Example Problem

Nozzle Shape: The nozzle is assumed to be of fixed shape. The following equation gives the desired properties of a supersonic convergent-divergent nozzle having a minimum area section.

$$A(x) = 1 + 2.2(x - 1.5)^2 \qquad 0 \le x \le 3$$

Note that x = 1.5 is the throat. $\left(\left. \frac{dA}{dx} \right|_{x=1.5} = 0 \right)$

In principle initial Conditions can be specified arbitrarily. However, steady state solution can be reached faster by judiciously choosing the initial conditions. An obvious choice would be the close-form relations between area ratio and the flow properties, which would almost be the same as the numerical solution that one would expect.

However, in order to test the robustness of a given numerical method, it would be preferable to choose the initial conditions that are not so close to the actual steady state solution. Anderson (1995) suggests the following linear variation.

$$\frac{\rho}{\rho_0} = 1 - 0.3146x$$

$$\frac{T}{T_0} = 1 - 0.2314x$$

$$\frac{u}{a_0} = (0.1 + 1.09x)\left(\frac{T}{T_0}\right)^{1/2}$$

where subscript 0 denotes reservoir conditions.

Program Completed

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