AE/ME 339
Computational Fluid Dynamics (CFD)

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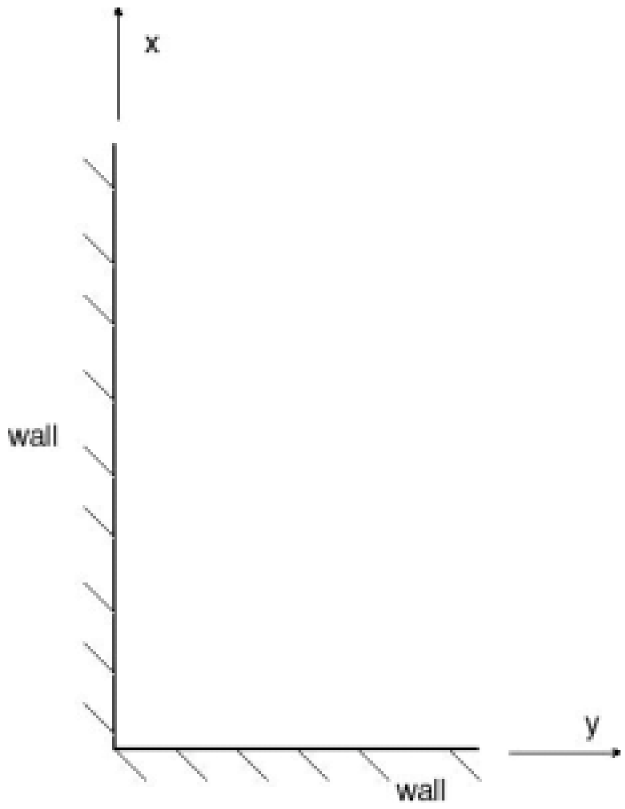
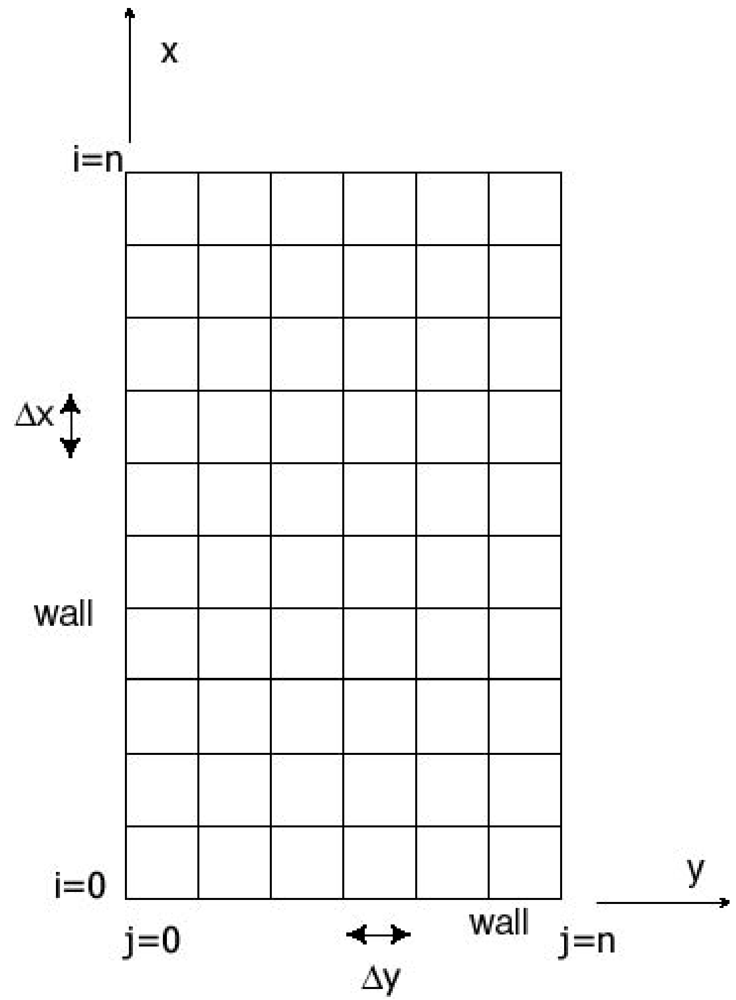
Let us consider flow over a heated wall as shown in figure.
A free convection flow induced by buoyancy forces is established close to the wall.

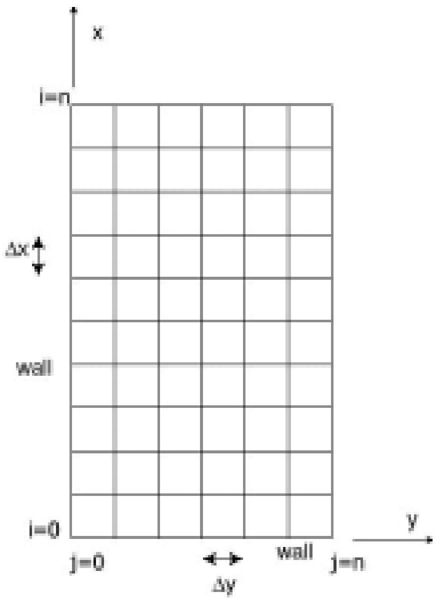



x
i=n
Δx
wall
i=0
j=0
Δy
wall
j=n
y

Boundary conditions:

At t = 0:
u = v = 0, T = T*inf* (everywhere in the solution domain)

At t > 0:
y = 0: u = v = 0, T = T*w* (heated wall conditior
y = inf: u = v = 0, T = T*inf* (far field condition)
x = 0: u = v = 0, T = T*inf* (bottom wall conditio



Governing equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \text{.............}(1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+g_x\beta(T-T_\infty)+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \text{.............}(2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+g_y\beta(T-T_\infty)+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \text{.............}(3)$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{\mu}{\rho c_p}\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2\right] \text{.............}(4)$$

The Navier-Stokes equations derived in Chapter 2 can be simplified using the boundary layer assumptions.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0..............(1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g_x\beta(T-T_\infty)+\nu\frac{\partial^2 u}{\partial y^2}..............(5)$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}..............(6)$$

Note the neglected terms in the above equations

Eq. 2: 2^{nd} derivative of u w.r.t x is neglected because it is small compared to the other term.
Pressure gradient is assumed to be zero.

Eq. 3: This equation is not solved in the present formulation
All the terms on the RHS are assumed to be zero.
Since p is assumed to be a constant, there is no need to use this equation.

Eq. 4: 2^{nd} derivative of temperature w.r.t x and the viscous dissipation terms are neglected.

The simplified equations (continuity, x-momentum and energy) can be non-dimensionalized as follows:

$$\xi = x(g\beta\Delta T/\nu^2)^{\frac{1}{3}}$$

$$\eta = y(g\beta\Delta T/\nu^2)^{\frac{1}{3}}$$

$$\tau = t(g\beta\Delta T)^{\frac{2}{3}}/\nu^{\frac{1}{3}}$$

$$\bar{u} = u/(\nu g\beta\Delta T)^{\frac{1}{3}}$$

$$\bar{v} = v/(\nu g\beta\Delta T)^{\frac{1}{3}}$$

$$\theta = \frac{T - T_\infty}{T_1 - T_\infty} = \frac{T - T_\infty}{\Delta T}$$

Note that g_x is the magnitude of thegravity component in the x-direction. β is the volumetric coefficient of thermal expansion defined as

Further, we assume that the Prandtl number Pr (= $\mu c_p/k = \nu/\alpha$) = 1. The equations will now be obtained in form that is easy to solve numerically.

$$\beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}$$

Note β has units of (1/K)

If the assumption Pr = 1 is not made, the RHS of the energy equation will have a Pr factor.

The energy equation for non-unity Prandtl number can be differenced using the explicit formulation as follows

$$\frac{\theta_{i,j}^{n+1}-\theta_{i,j}^{n}}{\Delta\tau}+\overline{u}_{i,j}^{n}\frac{\theta_{i,j}^{n}-\theta_{i-1,j}^{n}}{\Delta\xi}+\overline{v}_{i,j}^{n}\frac{\theta_{i,j+1}^{n}-\theta_{i,j}^{n}}{\Delta\eta}=$$

$$\frac{1}{\Pr}\left(\frac{\theta_{i,j+1}^{n}-2\theta_{i,j}^{n}+\theta_{i,j-1}^{n}}{(\Delta\eta)^{2}}\right)\quad\text{.................................}(7)$$

Note the use of BD for the second term on the LHS and FD for the third term and CD for the RHS term.

The x-momentum equation can now be written in the finite difference form as

$$\frac{\overline{u}^{n+1}_{i,j} - \overline{u}^{n}_{i,j}}{\Delta\tau} + \overline{u}^{n}_{i,j}\frac{\overline{u}^{n}_{i,j} - \overline{u}^{n}_{i-1,j}}{\Delta\xi} + \overline{v}^{n}_{i,j}\frac{\overline{u}^{n}_{i,j+1} - \overline{u}^{n}_{i,j}}{\Delta\eta}$$

$$= \theta^{n+1}_{i,j} + \frac{\overline{u}^{n}_{i,j+1} - 2\overline{u}^{n}_{i,j} + \overline{u}^{n}_{i,j-1}}{(\Delta\eta)^2} \qquad \text{.............................(8)}$$

The continuity equation becomes

$$\frac{\bar{u}_{i,j}^{n+1} - \bar{u}_{i-1,j}^{n+1}}{\Delta\xi} + \frac{\bar{v}_{i,j}^{n+1} - \bar{v}_{i,j-1}^{n+1}}{\Delta\eta} = 0 \text{...........(9)}$$

Equations 7, 8 and 9 are solved sequentially for each time step.
Note that θ*n+1* is known once the energy equation is solved.

Therefore, in the momentum equation, it can be treated as a known quantity.

Since u is known from solving the x-momentum equation, v*n+1* are the only unknowns in Eq. (9).

Thus these set of equations can be solved explicitly.

Program Completed

University of Missouri-Rolla