AE/ME 339 Computational Fluid Dynamics (CFD)

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The above vectors can be modified for the shock tube problem. Since only one space dimension is present, the vectors can be shortened as follows

$$\frac{\partial \bar{U}}{\partial t} + \frac{\partial \bar{F}}{\partial x} = \bar{J} \text{..................} (A)$$

$$\bar{U} = \begin{Bmatrix} \rho \\ \rho u \\ \rho \left(e + \dfrac{u^2}{2} \right) \end{Bmatrix} \text{...........} (B)$$

$$\bar{E} = \begin{Bmatrix} \rho u \\ \rho u^2 + p - \tau_{xx} \\ \rho\left(e + \frac{u^2}{2}\right)u + pu - k\frac{\partial T}{\partial x} - u\tau_{xx} \end{Bmatrix} \ldots\ldots\ldots\ldots\ldots (C)$$

$$\bar{J} = \begin{Bmatrix} 0 \\ 0 \\ \rho\dot{q} \end{Bmatrix} \ldots\ldots\ldots\ldots\ldots (D)$$

$$U_1 = \rho$$

$$U_2 = \rho u$$

$$U_3 = \rho(c_v T + \frac{u^2}{2})$$

$$u = U_2 / U_1$$

$$U_3 = U_1(c_v T + \frac{u^2}{2})$$

$$c_v T + \frac{u^2}{2} = \frac{U_3}{U_1}$$

$$T = \frac{1}{c_v}\left(\frac{U_3}{U_1} - \frac{u^2}{2}\right)$$

$$E_1 = \rho u$$

$$E_2 = \rho u^2 + p - \tau_{xx}$$

$$E_3 = \rho u(e + \frac{u^2}{2}) + pu + \dot{q}_x - u\tau_{xx}$$

Recall that, according to Stokes hypothesis $\lambda = -(2/3)\mu$

Also recall that

$$\tau_{xx} = \lambda \bar{\nabla} \cdot \bar{V} + 2\mu \frac{\partial u}{\partial x} = -\frac{2}{3}\mu \frac{\partial u}{\partial x} + 2\mu \frac{\partial u}{\partial x} = \frac{4}{3}\mu \frac{\partial u}{\partial x}$$

Therefore

$$E_2 = \rho u^2 + p - \frac{4}{3}\mu \frac{\partial u}{\partial x}$$

Also recall

$$E_1 = U_2$$

If we neglect external heating, the J-vector on the RHS will be = 0. Therefore we can write

$$\frac{\partial \overline{U}}{\partial t} = -\frac{\partial \overline{E}}{\partial x}$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial (\rho u)}{\partial x}$$

$$\frac{\partial (\rho u)}{\partial t} = -\frac{\partial (\rho u^2 + p - \frac{4}{3}\mu \frac{\partial u}{\partial x})}{\partial x}$$

$$\frac{\partial}{\partial t}\left[\rho (e + \frac{u^2}{2})\right] = -\frac{\partial}{\partial x}\left[\rho u (e + \frac{u^2}{2}) + pu - \frac{4}{3}\mu u \frac{\partial u}{\partial x}\right]$$

Auxiliary relations for a perfect gas

$$p = \rho R T$$

$$e = c_v T$$

$$h = c_p T$$

$$\frac{\overline{U}_i^{n+1} - U_i^n}{\Delta t} = -\frac{E_{i+1}^n - E_i^n}{\Delta x}$$

Calculate $\overline{E}^{n+1}$ from $\overline{U}^{n+1}$

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = -\frac{1}{2}\left[\frac{E_{i+1}^n - E_i^n}{\Delta x} + \frac{\overline{E}_i^{n+1} - \overline{E}_{i-1}^{n+1}}{\Delta x}\right]$$

$$\frac{U_i^{n+1}-U_i^n}{\Delta t}=-\frac{1}{2}\left[-\frac{\bar{U}_i^{n+1}-U_i^n}{\Delta t}+\frac{\bar{E}_i^{n+1}-\bar{E}_{i-1}^{n+1}}{\Delta x}\right]$$

$$U_i^{n+1}=U_i^n+\frac{1}{2}(\bar{U}_i^{n+1}-U_i^n)-\frac{\Delta t}{2\Delta x}(\bar{E}_i^{n+1}-\bar{E}_{i-1}^{n+1})$$

$$U_i^{n+1}=\frac{1}{2}(U_i^n+\bar{U}_i^{n+1})-\frac{\Delta t}{2\Delta x}(\bar{E}_i^{n+1}-\bar{E}_{i-1}^{n+1})$$

$$U_1 = \rho$$

$$U_2 = \rho u$$

$$U_3 = \rho(c_v T + \frac{u^2}{2})$$

$$E_1 = \rho u$$

$$E_2 = \rho u^2 + p - \frac{4}{3}\mu \frac{\partial u}{\partial x}$$

$$E_3 = \rho u(c_v T + \frac{u^2}{2}) + pu - \frac{4}{3}\mu u \frac{\partial u}{\partial x}$$

$$u = \frac{U_2}{U_1}$$

$$c_v T + \frac{u^2}{2} = \frac{U_3}{\rho}$$

$$T = \frac{U_3}{\rho c_v} - \frac{u^2}{2c_v} = \frac{1}{c_v}\left[\frac{U_3}{U_1} - \frac{1}{2}\frac{U_2^2}{U_1^2}\right] = \frac{1}{c_v U_1}\left[U_3 - \frac{1}{2}\frac{U_2^2}{U_1}\right]$$

$$p = \rho R T = U_1 R\left\{\frac{1}{c_v U_1}\left[U_3 - \frac{1}{2}\frac{U_2^2}{U_1}\right]\right\} = \frac{R}{c_v}\left[U_3 - \frac{1}{2}\frac{U_2^2}{U_1}\right]$$

$$E_1 = U_2$$

$$E_2 = u(\rho u) + p - \frac{4}{3}\mu\frac{\partial u}{\partial x}$$

$$E_2 = \frac{U_2}{U_1}U_2 + \frac{R}{c_v}U_3 - \frac{R}{2c_v}\frac{U_2^2}{U_1} - \frac{4}{3}\mu\frac{\partial u}{\partial x}$$

$$E_2 = \frac{R}{c_v}U_3 + \frac{U_2^2}{U_1}\left[1 - \frac{R}{2c_v}\right] - \frac{4}{3}\mu\frac{\partial u}{\partial x}$$

$$E_3 = U_2\left[\frac{c_p}{U_1 c_v}\left(U_3 - \frac{1}{2}\frac{U_2^2}{U_1}\right) + \frac{1}{2}\frac{U_2^2}{U_1^2}\right] - \frac{4}{3}\mu u\frac{\partial u}{\partial x}$$

The last terms in the momentum and energy equations that contain μ are the viscous dissipation terms.

The shock tube flow can be solved without including these terms (Euler form). However, the resulting numerical scheme will give rise to oscillations at sharp discontinuities such as the shock.

In order to avoid these oscillations, terms similar to the viscous terms are introduced to smear out the discontinuities.
These terms are sometimes called the “artificial viscosity term.”

von Neumann and Richtmyer used a form of the artificial viscosity term which for the MacCormack scheme can be written as follows.

von Neumann-Richtmyer artificial viscosity term and its two finite difference forms are given below.

$$D\frac{\partial u}{\partial x} = \alpha \Delta x \rho \left|\begin{matrix}0\\1\\u\end{matrix}\right| \left|\frac{\partial u}{\partial x}\right| \frac{\partial u}{\partial x}$$

$$D_{i+\frac{1}{2}}(u_{i+1}-u_i) = \alpha \rho_{i+\frac{1}{2}} \left|\begin{matrix}0\\1\\u\end{matrix}\right|_{i+\frac{1}{2}} \left|u_{i+1}-u_i\right|(u_{i+1}-u_i)$$

$$D_{i+\frac{1}{2}}(u_{i+1}-u_i) = \alpha \rho_i \left|\begin{matrix}0\\1\\u\end{matrix}\right|_i \left|u_{i+1}-u_i\right|(u_{i+1}-u_i)$$

Note that D is a positive quantity. α is an empirical factor of $O(1)$ to be adjusted by closely monitoring the solution. α should be as small as possible and yet avoid oscillations.

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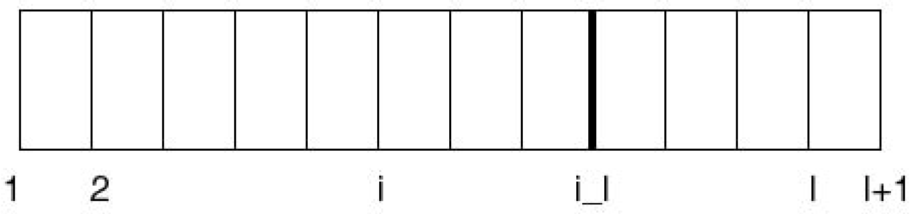
$$\bar{U}_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}(E_{i+1}^n - E_i^n) +$$

$$\frac{\Delta t}{\Delta x}\left[D_{i+\frac{1}{2}}^n (U_{i+1}^n - U_i^n) - D_{i-\frac{1}{2}}^n (U_i^n - U_{i-1}^n) \right]$$

$$U_i^{n+1} = \frac{1}{2}(U_i^n + \bar{U}_i^{n+1}) - \frac{\Delta t}{2\Delta x}(\bar{E}_i^{n+1} - \bar{E}_{i-1}^{n+1}) +$$

$$\frac{\Delta t}{\Delta x}\left[\bar{D}_{i+\frac{1}{2}} (\bar{U}_{i+1} - \bar{U}_i) - \bar{D}_{i-\frac{1}{2}} (\bar{U}_i - \bar{U}_{i-1}) \right]$$



Shock Tube Schematic

Initial conditions

$$u_i^n = 0 \ldots\ldots\ldots i = 1, I+1$$

$$p_i^n = p_1 \ldots\ldots\ldots i = 1, i_l$$

$$p_i^n = p_2 \ldots\ldots\ldots i = i_l + 1, I+1$$

$$T_i^n = T_0 \ldots\ldots\ldots i = 1, I+1$$

Boundary conditions

Walls at I = 1 and I = I+1

$$u_1 = 0$$

$$u_{I+1} = 0$$

Open ends

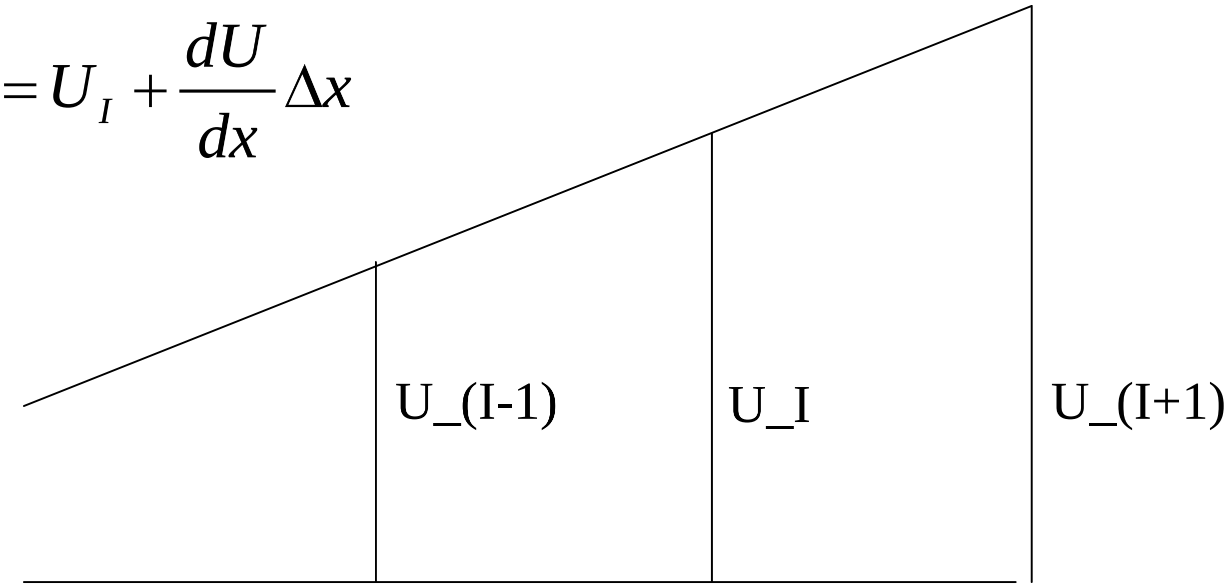
$$p_1 = p_{amb}$$

$$p_{I+1} = p_{amb}$$

Other boundary conditions can be obtained by linear extrapolation

$$\frac{dU}{dx} = \frac{U_I - U_{I\_1}}{\Delta x}$$

$$U_{I+1} = U_I + \frac{dU}{dx}\Delta x$$



Substitution gives

$$U_{I+1}^{n+1} = 2U_{I}^{n} - U_{I-1}^{n}$$

Similarly an expression can be written for the left boundary

$$\frac{dU}{dx} = \frac{U_3 - U_2}{\Delta x}$$

$$U_1 = U_2 - \frac{dU}{dx}\Delta x$$

$$U_1^{n+1} = 2U_2^{n} - U_3^{n}$$