AE/ME 339
Computational Fluid Dynamics (CFD)

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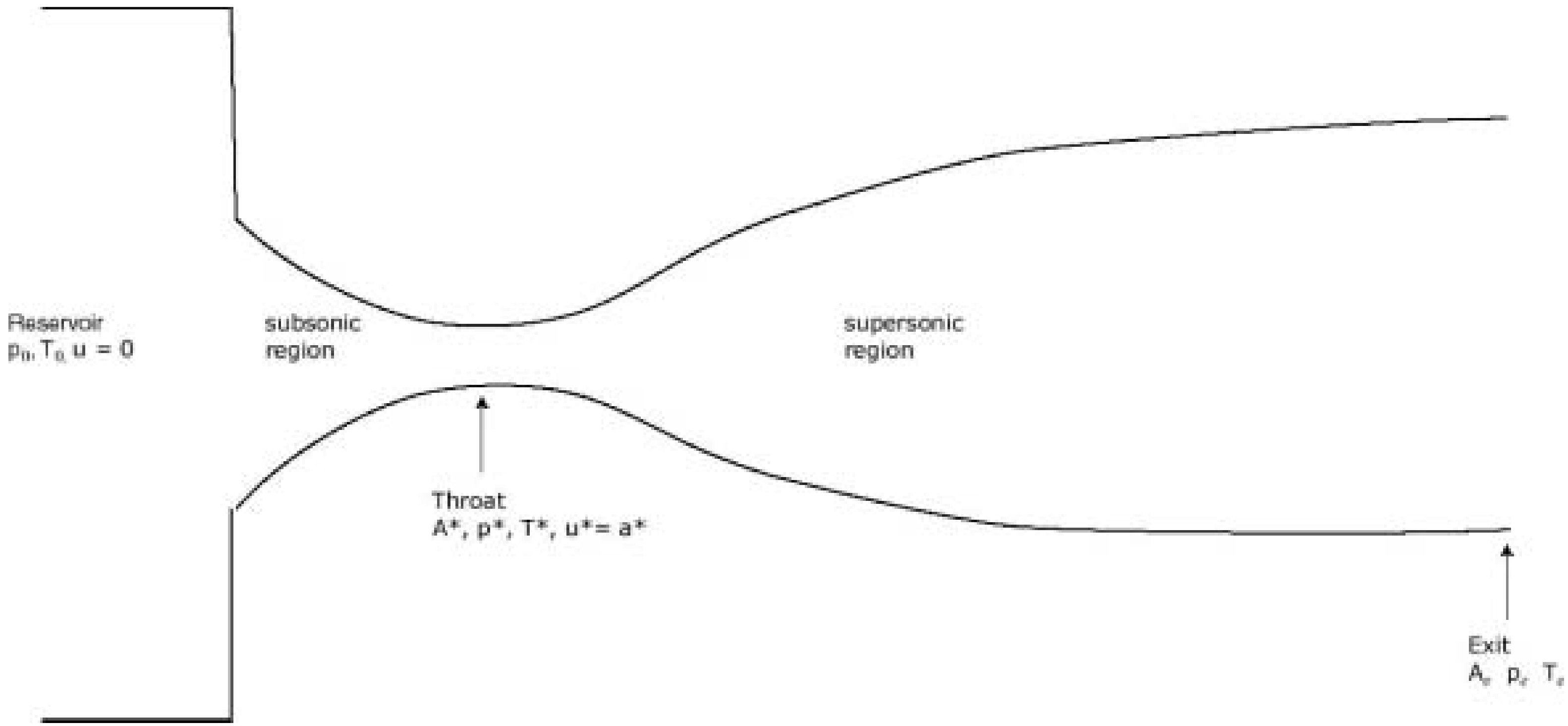
Quasi One-Dimensional Nozzle Flow



Consider the steady isentropic flow through a convergent-divergent nozzle shown in the Figure 19.1

The nozzle is attached to a large reservoir in which the conditions remains constant, denoted by pressure p_0 and temperature T_0.

p_0 and T_0 are known as the stagnation pressure and stagnation temperature, respectively.

If the nozzle exhausting into a region in which the pressure is much smaller (exact pressure ratios will be established as we proceed) compared to the reservoir pressure.

The flow can be analyzed as a quasi one-dimensional flow (flow properties are a function of the axial location only; there is no variation in the radial or azimuthal direction).

Under the right conditions, the flow accelerates gradually along the nozzle axis and attains a Mach number greater than one at the exit.
The flow variables are given by close form solutions which can be used to compare any numerical results.

Isentropic one-dimensional flow

We assume that the flow variables can be represented as a function of the axial coordinate alone.

For steady flow we can write

$$\rho \mathrm{Au} = \text{constant}$$

where A is the nozzle cross-sectional area

Taking the logrithm and differentiating yields

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{du}{u} = 0$$

The momentum equation can be written as

$$d\frac{u^2}{2} + \frac{dp}{\rho} = 0$$

Rewriting

$$udu + \frac{d\rho}{\rho}\frac{dp}{d\rho} = udu + a^2\frac{d\rho}{\rho} = 0$$

substitute for $\frac{d\rho}{\rho}$ from continuity equation

$$udu - a^2\left(\frac{du}{u} + \frac{dA}{A}\right) = 0$$

$$\frac{dA}{A} = \frac{u^2}{a^2}\frac{du}{u} - \frac{du}{u} = \frac{du}{u}(M^2 - 1)$$

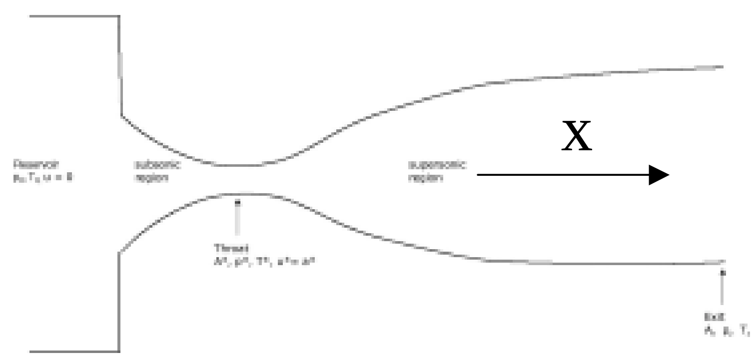
The above equation can be rewritten as

$$\frac{1}{A}\frac{dA}{dx} = \left(\frac{M^2 - 1}{u}\right)\frac{du}{dx}$$

This equation can be used to discuss nozzle shape

Case 1:

subsonic region (M < 1, $(M^2 - 1) < 0$)



If the passge is converging $(\frac{dA}{dx} < 0)$, then the flow will accelerate because $\frac{du}{dx} > 0$.

Similar arguments indicate that when the Mach number is > 1, the flow will accelerate in a diverging passage. The opposite will be true for decelerating flow.

Adiabatic relation for an ideal gas

$$\frac{T_0}{T} = \left[1 + \frac{\gamma - 1}{2} M^2\right]$$

Isentropic relations for an ideal gas

$$\frac{p_0}{p} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{\gamma}{\gamma - 1}}$$

$$\frac{\rho_0}{\rho} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{1}{\gamma - 1}}$$

Equations of Nozzle Flow.

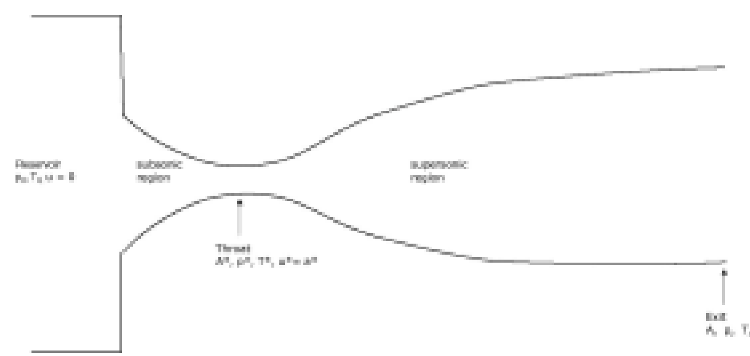
Mass flow rate: $\dot{m} = \rho A u$

Can be transformed using isentropic relation as:

$$\dot{m} = \rho_0 \; uA \frac{\rho}{\rho_0} = \rho_0 \; uA \left(\frac{p}{p_0} \right)^{1/\gamma}$$

Substitution for the throat conditions gives

$$\dot{m} = \frac{A^* p_0}{R T_0} \sqrt{\gamma \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma - 1}{\gamma}}}$$



The area ratio can be expressed in terms of the Mach number as follows

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$

Note that the area ratio between any given location in the nozzle and the throat is only a function of the Mach number. Once the area ratio is known, the corresponding Mach number can be determined.

Note also that the flow must be isentropic (i. e., no shocks present in the nozzle passage) for these results to hold.

Insert Figure 7.2

CFD Solution of Nozzle Flow

Figure 7.3

Continuity: $$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A u) = 0$$

Momentum: $$\frac{\partial}{\partial t}(\rho u A) + \frac{\partial}{\partial x}(\rho u^2 A) = -A\frac{\partial p}{\partial x}$$

Energy: $$\frac{\partial}{\partial t}(\rho e A) + \frac{\partial}{\partial x}(\rho e u A) = -p\frac{\partial}{\partial x}(Au)$$

Equations in non-conservation form for ideal gas

Continuity:
$$\frac{\partial}{\partial t}(\rho A) + \rho A \frac{\partial u}{\partial x} + uA \frac{\partial \rho}{\partial x} = 0$$

Momentum:
$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = -R\left(\rho \frac{\partial T}{\partial x} + T \frac{\partial \rho}{\partial x} \right)$$

Energy:
$$\rho \mathrm{c_v} \frac{\partial T}{\partial t} + \rho u \mathrm{c_v} \frac{\partial T}{\partial x} = -\rho RT \left[\frac{\partial u}{\partial x} + u \frac{\partial (\ln A)}{\partial x} \right]$$

The finite difference form

We divide the nozzle axis (x-axis) into a number of segments, I, each of length Δx. Therefore, beginning with first node (1), we will have a total of (I+1) nodes. Nodes 1 and (I+1) are the boundary nodes. MacCormack's method can now be used for the numerical solution.

Applying MacCormack's method to the non-conservation form yields the following:

$$\left(\frac{\partial \rho}{\partial t}\right)_i^n = -\rho_i^n \left(\frac{u_{i+1}^n - u_i^n}{\Delta x}\right) - \rho_i^n u_i^n \left(\frac{\ln A_{i+1} - \ln A_i}{\Delta x}\right) - u_i^n \left(\frac{\rho_{i+1}^n - \rho_i^n}{\Delta x}\right)$$

$$\left(\frac{\partial u}{\partial t}\right)_i^n = -u_i^n \left(\frac{u_{i+1}^n - u_i^n}{\Delta x}\right) - R\left(\frac{T_{i+1} - T_i}{\Delta x}\right) - R\left(\frac{T}{\rho}\right)_i^n \left(\frac{\rho_{i+1}^n - \rho_i^n}{\Delta x}\right)$$

$$\left(\frac{\partial T}{\partial t}\right)_i^n = -u_i^n \left(\frac{T_{i+1}^n - T_i^n}{\Delta x}\right) - (\gamma - 1)T_i^n \left[\frac{u_{i+1}^n - u_i^n}{\Delta x} + u_i^n \frac{\ln A_{i+1}^n - \ln A_i^n}{\Delta x}\right]$$

Predictor Step

$$\bar{\rho}_i^{n+1} = \rho_i^n + \left(\frac{\partial \rho}{\partial t}\right)_i^n \Delta t$$

$$\bar{u}_i^{n+1} = u_i^n + \left(\frac{\partial u}{\partial t}\right)_i^n \Delta t$$

$$\bar{T}_i^{n+1} = T_i^n + \left(\frac{\partial T}{\partial t}\right)_i^n \Delta t$$

$$\left(\frac{\partial \bar{\rho}}{\partial t}\right)_i^{n+1} = -\bar{\rho}_i^n \left(\frac{\bar{u}_i^n - \bar{u}_{i-1}^n}{\Delta x}\right) - \bar{\rho}_i^n \bar{u}_i^n \left(\frac{\ln A_i - \ln A_{i-1}}{\Delta x}\right) - \bar{u}_i^n \left(\frac{\bar{\rho}_i^n - \bar{\rho}_{i-1}^n}{\Delta x}\right)$$

$$\left(\frac{\partial \bar{u}}{\partial t}\right)_i^{n+1} = -\bar{u}_i^n \left(\frac{\bar{u}_i^n - \bar{u}_{i-1}^n}{\Delta x}\right) - R\left(\frac{\bar{T}_i - \bar{T}_{i-1}}{\Delta x}\right) - R\left(\frac{\bar{T}}{\bar{\rho}}\right)_i^n \left(\frac{\bar{\rho}_i^n - \bar{\rho}_{i-1}^n}{\Delta x}\right)$$

$$\left(\frac{\partial \bar{T}}{\partial t}\right)_i^{n+1} = -\bar{u}_i^n \left(\frac{\bar{T}_i^n - \bar{T}_{i-1}^n}{\Delta x}\right) - (\gamma - 1)\bar{T}_i^n \left[\frac{\bar{u}_i^n - \bar{u}_{i-1}^n}{\Delta x} + \bar{u}_i^n \frac{\ln A_i^n - \ln A_{i-1}^n}{\Delta x}\right]$$

Find average of the derivatives

$$\left(\frac{\partial \rho}{\partial t}\right)_{av} = \frac{1}{2}\left[\left(\frac{\partial \rho}{\partial t}\right)_i^n + \left(\frac{\partial \overline{\rho}}{\partial t}\right)_i^{\mathrm{n+1}}\right]$$

$$\left(\frac{\partial u}{\partial t}\right)_{av} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)_i^n + \left(\frac{\partial \overline{u}}{\partial t}\right)_i^{n+1}\right]$$

$$\left(\frac{\partial T}{\partial t}\right)_{av} = \frac{1}{2}\left[\left(\frac{\partial T}{\partial t}\right)_i^n + \left(\frac{\partial \overline{T}}{\partial t}\right)_i^{n+1}\right]$$

Corrected values, time level (n+1)

$$\rho_i^{n+1} = \rho_i^n + \left(\frac{\partial \rho}{\partial t} \right)_{av}^{n+1} \Delta t$$

$$u_i^{n+1} = u_i^n + \left(\frac{\partial u}{\partial t} \right)_{av}^{n+1} \Delta t$$

$$T_i^{n+1} = T_i^n + \left(\frac{\partial T}{\partial t} \right)_{av}^{n+1} \Delta t$$

Courant Number $C = (u + a)\dfrac{\Delta t}{\Delta x}$, can be used to determine the time step that satisfies the stability criterion. $C \leq 1$ ensures that the solution will be stable. Note the above stability criterion is an empirical relation and therefore, it would be safe to use a time step for which the value of C will be well below unity.

Boundary conditions

The boundary conditions can be established using the theory of the method of characteristics (Tannehill, et al. 1997).

At the subsonic inflow boundary we set the values of T and ρ and use linear extrapolation for u.

At the nozzle exit, T, ρ, and u are obtained by using linear extrapolation.

Equations in conservation form

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A u) = 0$$

$$\frac{\partial}{\partial t}(\rho A u) + \frac{\partial}{\partial x}(\rho A u^2 + pA) = p\frac{\partial A}{\partial x}$$

$$\frac{\partial}{\partial t}\left(\rho A(e + \frac{u^2}{2})\right) + \frac{\partial}{\partial x}\left(\rho A(e + \frac{u^2}{2})u + pAu\right) = 0$$

The above equations can be solved in the same manner as the shock tube flow equations.

Nozzle Shape: The nozzle is assumed to be of fixed shape. The following equation gives the desired properties of a supersonic convergent-divergent nozzle having a minimum area section.

$$A(x) = 1 + 2.2(x - 1.5)^2 \qquad 0 \le x \le 3$$

Note that x = 1.5 is the throat. $\left(\left. \frac{dA}{dx} \right|_{x=1.5} = 0 \right)$

In principle initial Conditions can be specified arbitrarily. However, steady state solution can be reached faster by judiciously choosing the initial conditions. An obvious choice would be the close-form relations between area ratio and the flow properties, which would almost be the same as the numerical solution that one would expect.

However, in order to test the robustness of a given numerical method, it would be preferable to choose the initial conditions that are not so close to the actual steady state solution. Anderson (1995) suggests the following linear variation

$$\frac{\rho}{\rho_0} = 1 - 0.3146x$$

$$\frac{T}{T_0} = 1 - 0.2314x$$

$$\frac{u}{a_0} = (0.1 + 1.09x)\left(\frac{T}{T_0}\right)^{1/2}$$

where subscript 0 denotes reservoir conditions.