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Course Information

Course Outline

- Ordinary differential equations (ODE)
- Numerical techniques for solving ODEs
- Example: Flow in constant area pipe with heat addition and friction
- Partial differential equations, classification
- Discretization of derivatives
- Errors and analysis of stability
- Example: Unsteady heat conduction in a rod
- Example: Natural convection at a heated vertical plate
- Discretization techniques

Course Outline (continued)

- Couette flow
- The shock tube problem
- Introduction to packaged codes:
  - Grid generation
  - Problem setup
  - Solution
- Turbulence modeling

ODEs and PDEs may be discretized-approximated-
as a set of algebraic equations and solved

Discretization methods for ODEs are well known

e.g., Runge-Kutta methods for initial value problems
and shooting methods for BV problems

PDEs involve more than 1 independent variable
e.g., x, y, z, t in Cartesian coordinates for time-dependent
Problems

PDEs can be discretized using finite difference
Methods

PDEs can also be discretized in integral form, known as finite volume methods

Sometimes coordinate transformation is necessary before discretization

Flow with heat addition and friction Ref: Hill & Peterson, Mechanics and Thermodynamics of Propulsion, Addison-Wesley

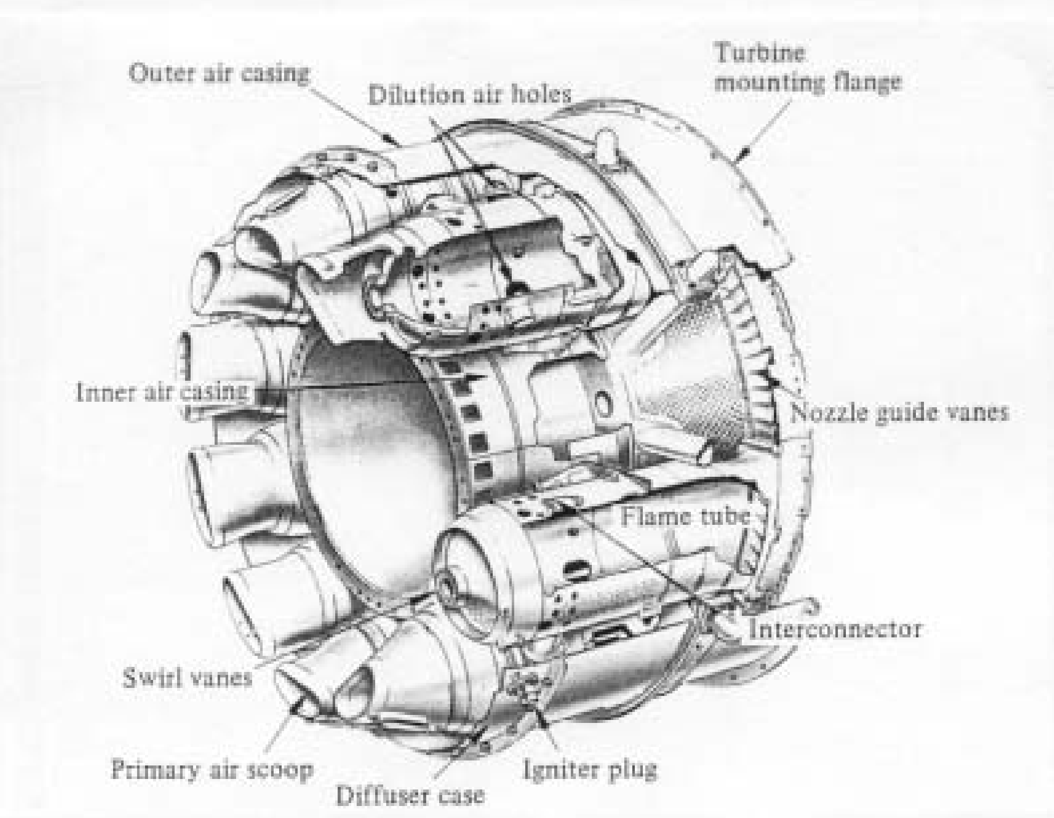


FIGURE 6.28 Can-annular combustion chamber. (Courtesy Rolls-Royce, plc.)

Flow with heat addition and friction Ref: Hill & Peterson, Mechanics and Thermodynamics of Propulsion, Addison-Wesley

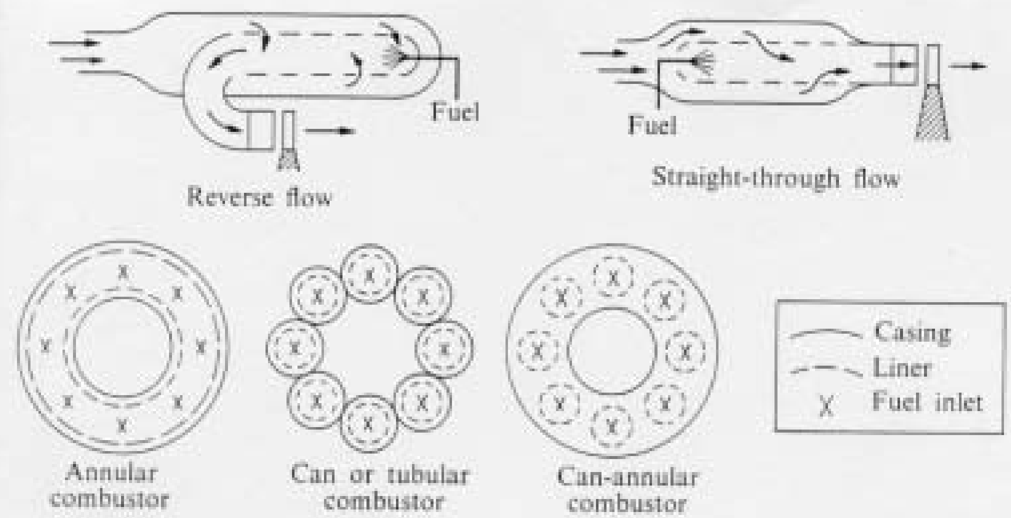


FIGURE 6.27 Combustion chamber, longitudinal and cross sections.

Flow with heat addition and friction Ref: Hill & Peterson, Mechanics and Thermodynamics of Propulsion, Addison-Wesley

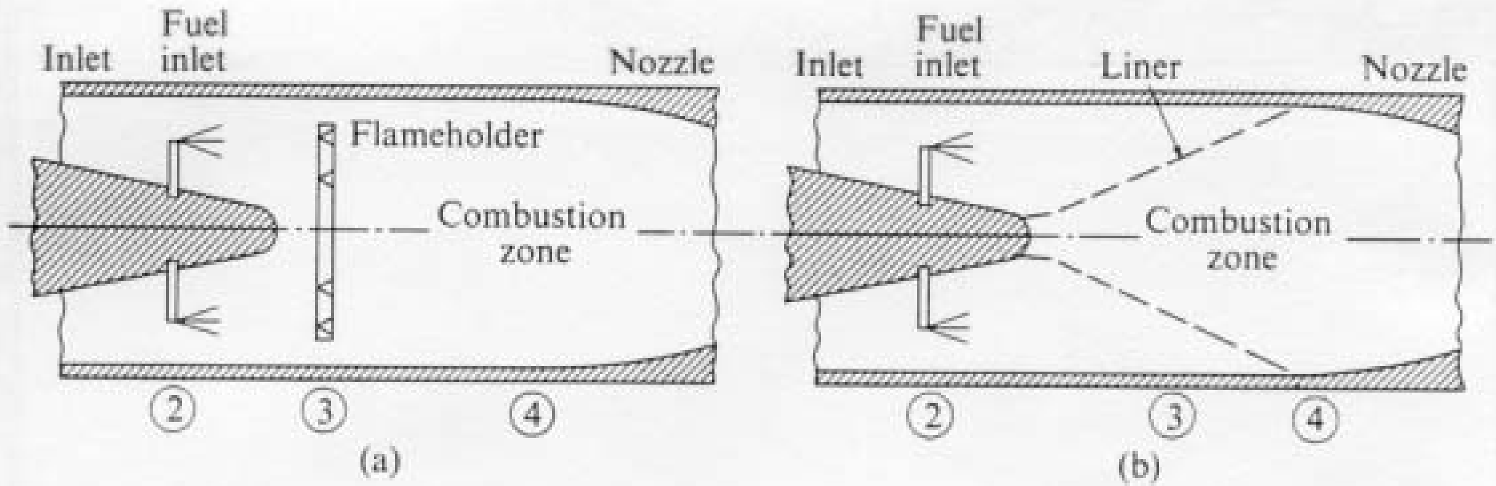


FIGURE 6.35 Schematic ramjet (or afterburner) combustion chambers.

Flow with heat addition and friction

Ref: Hill & Peterson, Mechanics and Thermodynamics of Propulsion, Addison-Wesley

p_1, ρ_1, T_1, u_1

p_2, ρ_2, T_2, u_2

Flow with heat addition and friction

Perfect gas flows from left to right in a constant area duct
Heat addition and/or friction may be present
Flow properties will change during the process
Equations can be solved analytically when either heat addition or friction is present, but not both

Flow with heat addition

Stagnation enthalpy change (Conservation of Energy/First law of thermodynamics)

$$\Delta h_0 = q - w \qquad (1)$$

Conservation of mass

$$\frac{d\rho}{\rho} + \frac{du}{u} = 0 \qquad (2)$$

Flow with heat addition

Momentum

$$dp = -\rho u du \qquad (3)$$

Stagnation Enthalpy

$$dh_0 = dh + u du \qquad (4)$$

Flow with heat addition

Integration yields

$$\rho_1 u_1 = \rho_2 u_2$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \qquad (5)$$

Equation of state $p = \rho RT$

Speed of sound $a^2 = \gamma RT$

Flow with heat addition

Above equations can be combined to yield the following

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}$$

$$\frac{p_{02}}{p_{01}} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}}$$

Flow with heat addition

$$\frac{T_2}{T_1} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2} \frac{M_2}{M_1} \right)^2$$

The above equation + the following adiabatic flow relation can be used to get stagnation temperature ratio

$$\frac{T_0}{T} = \left(1 + \frac{\gamma - 1}{2} M^2 \right)$$

Flow with heat addition

$$\frac{T_{02}}{T_{01}} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2} \frac{M_1}{M_1} \right)^2 \left(\frac{1+0.5(\gamma-1)M_2^2}{1+0.5(\gamma-1)M_1^2} \right)$$

Note: Final Mach number depends on initial Mach number and final stagnation temperature.

Flow with heat addition

Reference conditions
Note that the stagnation conditions change due to heat addition
For given initial conditions, Mach 1 conditions, denoted by (*) can be used for reference
Thus

$$\frac{T}{T^*} = M^2 \left[\frac{1+\gamma}{1+\gamma M^2} \right]^2$$

Flow with heat addition

$$\frac{T_0}{T_0^*} = \frac{2(\gamma+1)M^2\left[1+0.5(\gamma-1)M^2\right]}{\left[1+\gamma M^2\right]^2}$$

$$\frac{p}{p^*} = \frac{1+\gamma}{1+\gamma M^2}$$

$$\frac{p_0}{p_0^*} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{1+\gamma}{1+\gamma M^2}\right)\left(1+\frac{(\gamma-1)}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

The above equation shows that, for given initial conditions, fluid properties are only a function of the local Mach number

Flow with heat addition

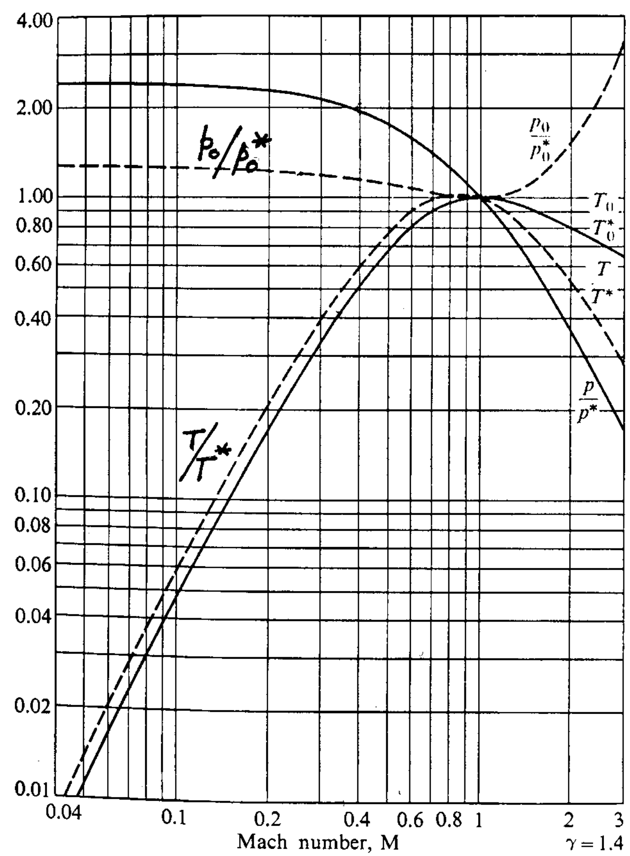


FIGURE 3.4 Frictionless flow of a perfect gas in a constant-area duct with stagnation temperature change. (From Shapiro [1].)

Flow with heat addition

Calculation procedure, given p_{01}, T_{01}, M_1, and q
Determine T_0^* using T_{01} and M_1
Determine T_{02}/T_0^* using

$$\frac{T_{02}}{T_0^*} = \frac{T_{01}}{T_0^*} + \frac{\Delta T_0}{T_0^*} = \frac{T_{01}}{T_0^*} + \frac{q}{c_p T_0^*}$$

Calculate M_2, p_2, p_{02}

Flow with heat addition

Observe in figure, for subsonic and supersonic cases
Heat addition drives M towards 1. Results in "thermal choking." There is a loss of stagnation pressure.

Flow with heat addition and friction

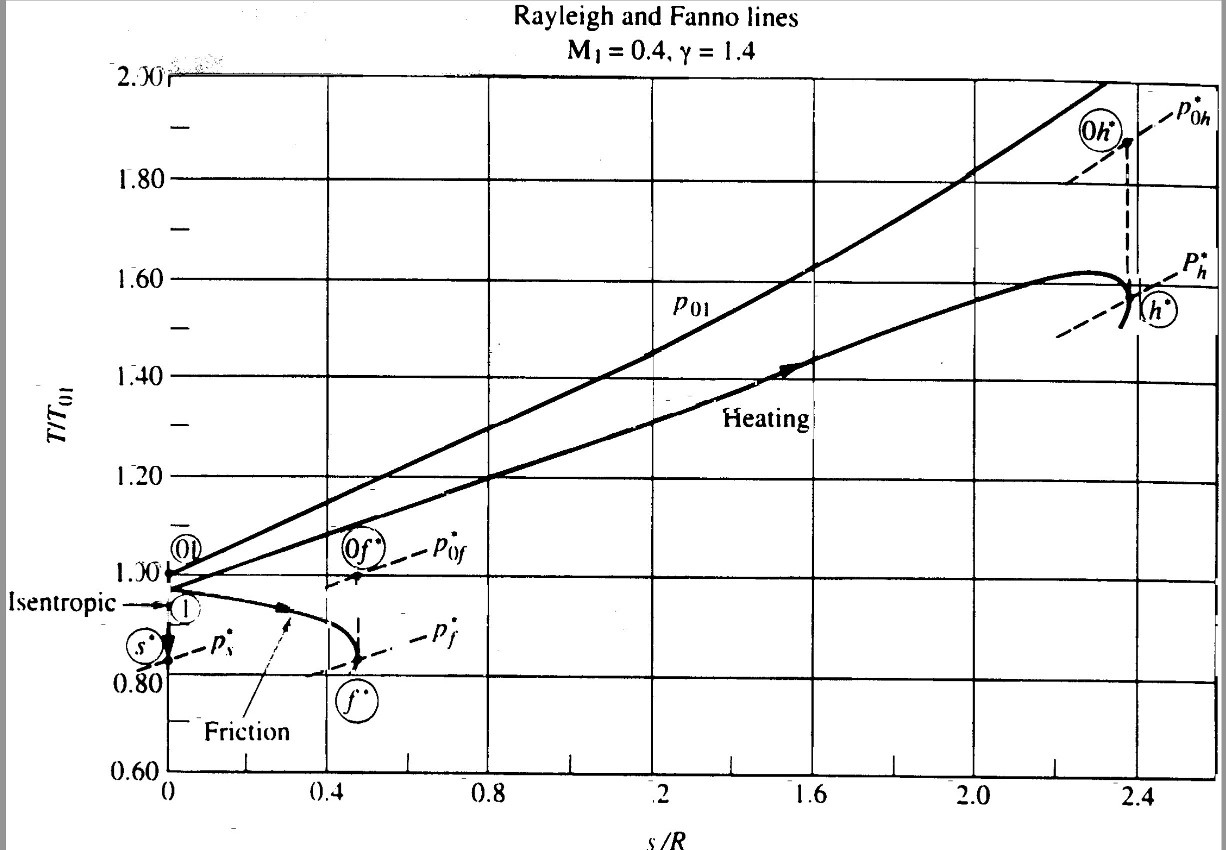


FIGURE 3.6 Typical temperature–entropy diagram for the choking process (subsonic).

Flow with heat addition and friction

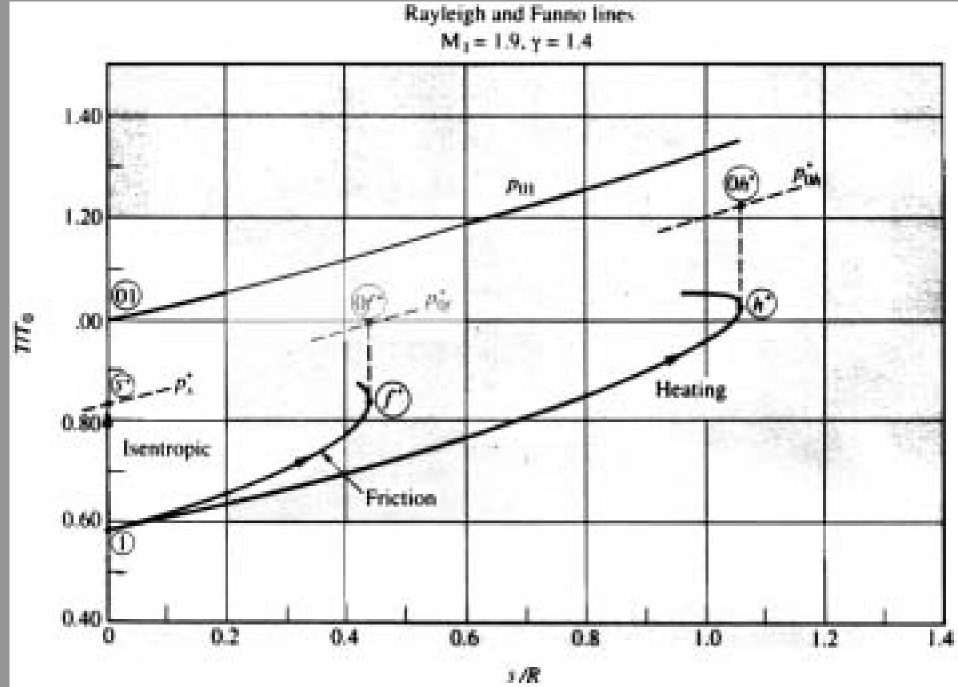


FIGURE 3.7 Typical temperature–entropy diagram for the choking process (supersonic)

Flow with friction

Momentum equation

$$dp + \rho u du + \frac{\tau_0 c dx}{A} = 0$$

where c is the circumference and A is the cross-section area. cdx is the curved surface area of the tube of length dx. τ_0 is the wall shear stress (N/m^2)

Flow with friction

The energy equation in this case is

$$h_0 = h + u^2/2 = \text{constant}$$

or $$dh + udu = 0$$

Shear stress correlation for fully developed pipe flow

$$c_f = \frac{\tau_0}{\rho u^2 / 2} = f\left(\frac{\rho u d}{\mu}, \frac{\varepsilon}{d}\right)$$

Flow with friction

where ε is the rms roughness of the pipe wall
c_f is the skin coefficient

The equations of continuity, momentum and energy can now be combined with the perfect gas equation of state to get the equations for flow with friction

Flow with friction

See Hill & Peterson for detailed derivation of the following equation

$$\frac{dM^2}{M^2} = \frac{\gamma M^2 \left(1 + \frac{(\gamma - 1)}{2} M^2\right)}{1 - M^2} \left(\frac{4 c_f dx}{D}\right)$$

Note the behavior of the flow for subsonic and supersonic cases. In both cases, Mach number tends towards 1. Condition is called friction choking

Flow with friction

Integrating and applying the limit between M = M and M = 1 yields the following result for "length to choke," L^*

$$\frac{4c_f L^*}{D} = \frac{1-M^2}{\gamma M^2} + \frac{\gamma+1}{2\gamma}\ln\left[\frac{(\gamma+1)M^2}{2\left(1+\frac{(\gamma-1)}{2}M^2\right)}\right]$$

Flow with friction

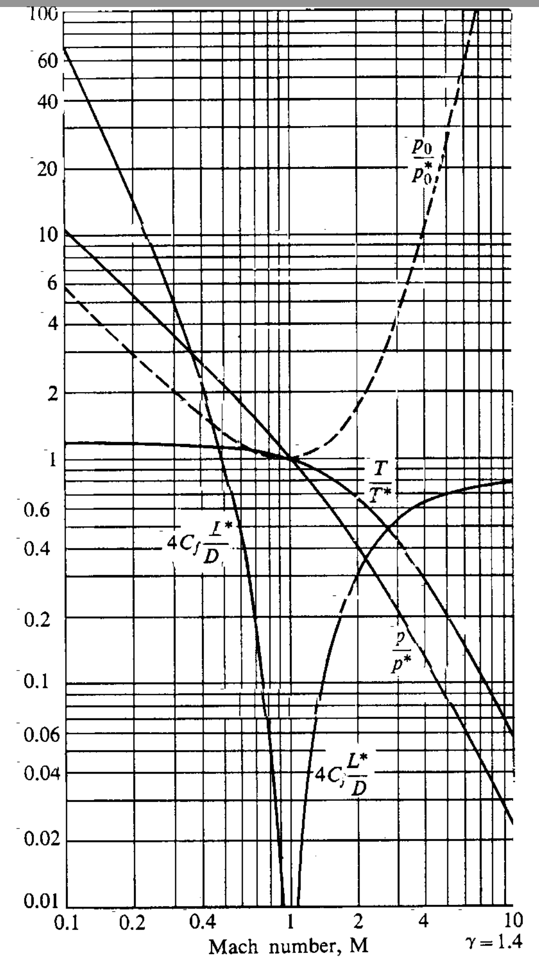


FIGURE 3.5 Adiabatic flow of a perfect gas in a constant-area duct with friction. (From Shapiro [1].)

Flow with heat addition and friction

The continuity equation is the same as before.
The momentum equation has the shear stress term.
The energy equation has the heat addition (q) term.
These equations can now be combined with the perfect gas equation to get the differential equation which does not have a closed-form solution.
Solution can be obtained by numerical integration.

$$\frac{d\rho}{\rho} + \frac{du}{u} = 0 \qquad (1)$$

$$\frac{dp}{p} = \frac{-\tau_0 c dx}{pA} - \frac{\rho u du}{p} \qquad (2)$$

$$c_p dT_0 = c_p dT + u du$$

$$dT_0 = dT + \frac{(\gamma - 1) u du}{\gamma R} \qquad (3)$$

$$M = \frac{u}{\sqrt{\gamma RT}}, \frac{dM^2}{M^2} = \frac{du^2}{u^2} - \frac{dT}{T} \qquad (4)$$

$$p = \rho RT$$

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

$$\left[\frac{d\rho}{\rho} = \frac{dp}{p} - \frac{dT}{T}\right] \qquad (5)$$

Momentum

$$\frac{dp}{p} = \frac{-M^2\gamma}{2}\left(\frac{4c_f dx}{D}\right) - \frac{udu}{RT} \qquad (6)$$

From (1)

$$\frac{d\rho}{\rho}+\frac{1}{2}\frac{du^2}{u^2}=0$$

$$\frac{du^2}{u^2}=-2\frac{d\rho}{\rho}=-2\left[\frac{dp}{p}-\frac{dT}{T}\right]$$

$$\frac{du^2}{u^2}=-2\frac{dp}{p}+2\frac{dT}{T} \qquad (7)$$

Substitute in (4)

$$\frac{dM^2}{M^2}=-2\frac{dp}{p}+2\frac{dT}{T}-\frac{dT}{T}=-2\frac{dp}{p}+\frac{dT}{T} \qquad (8)$$

Combine (3) and (6)

$$\frac{dp}{p} = -\frac{\gamma M^2}{2}\left(\frac{4c_f dx}{D}\right) - \frac{c_p dT_0}{RT} + \frac{c_p dT}{RT}$$

$$\frac{dp}{p} = -\frac{\gamma M^2}{2}\left(\frac{4c_f dx}{D}\right) - \frac{\gamma}{(\gamma - 1)}\frac{dT_0}{T} + \frac{\gamma}{(\gamma - 1)}\frac{dT}{T} \qquad (9)$$

$$\frac{p_0}{p} = \left[1 + \frac{(\gamma - 1)}{2} M^2\right]^{\frac{\gamma}{(\gamma - 1)}}$$

$$\frac{dp_0}{p_0} - \frac{dp}{p} = \frac{\dfrac{\gamma}{(\gamma - 1)} \dfrac{(\gamma - 1)}{2} dM^2}{\left[1 + \dfrac{(\gamma - 1)}{2} M^2\right]} = \frac{\gamma}{2} \frac{dM^2}{\left[1 + \dfrac{(\gamma - 1)}{2} M^2\right]} \qquad (10)$$

$$\frac{T_0}{T} = \left[1 + \frac{(\gamma - 1)}{2} M^2\right]$$

$$\frac{dT_0}{T_0} - \frac{dT}{T} = \frac{\frac{(\gamma - 1)}{2} dM^2}{\left[1 + \frac{(\gamma - 1)}{2} M^2\right]} \qquad (11)$$

Substitute (10) and (11) in (9)

$$\frac{dp_0}{p} - \frac{\gamma}{2} \frac{dM^2}{\left[1 + \frac{(\gamma - 1)}{2} M^2\right]} = -\frac{\gamma M^2}{2}\left(\frac{4c_f\, dx}{D}\right)$$

$$-\frac{\gamma}{(\gamma - 1)}\left[\frac{dT_0}{T_0}\frac{T_0}{T} - \frac{dT}{T}\right]$$

2nd RHS term:

$$= \frac{-\gamma}{(\gamma - 1)}\left[\frac{dT_0}{T_0}\left(1 + \frac{(\gamma - 1)}{2} M^2\right) - \frac{dT_0}{T_0} + \frac{\frac{(\gamma - 1)}{2} dM^2}{1 + \frac{(\gamma - 1)}{2} M^2}\right]$$

$$\frac{dp_0}{p_0} = -\frac{M^2\gamma}{2}\left(\frac{4c_f dx}{D}\right) - \frac{dT_0}{T_0}\frac{\gamma}{(\gamma-1)}\left[1+\frac{(\gamma-1)}{2}M^2-1\right]$$

$$\frac{dp_0}{p_0} = -\frac{M^2\gamma}{2}\left(\frac{4c_f dx}{D}\right) - \frac{\gamma}{2}M^2\frac{dT_0}{T_0}$$

$$\frac{dp_0}{p_0} = -\frac{M^2\gamma}{2}\left[\frac{dT_0}{T_0} + \frac{4c_f dx}{D}\right] \qquad (12)$$

$$\frac{dT}{T}=\frac{dT_0}{T_0}-\frac{\frac{(\gamma-1)}{2}dM^2}{1+\frac{(\gamma-1)}{2}M^2} \qquad (11)$$

Substitute in (8) using (10) and (11)

$$\frac{dM^2}{M^2}=-2\frac{dp_0}{p_0}+\frac{dM^2\gamma}{1+\frac{(\gamma-1)}{2}M^2}+\frac{dT_0}{T_0}-\frac{\frac{(\gamma-1)}{2}dM^2}{1+\frac{(\gamma-1)}{2}M^2}$$

$$\frac{dM^2}{M^2}\left[1-\frac{M^2\left(\gamma-\frac{(\gamma-1)}{2}\right)}{1+\frac{(\gamma-1)}{2}M^2}\right]=\left[-2\frac{dp_0}{p_0}+\frac{dT_0}{T_0}\right]$$

LHS factor:

$$\frac{\left(1+\frac{(\gamma-1)M^2}{2}-M^2\left(\frac{\gamma+1}{2}\right)\right)}{\left(1+\frac{\gamma-1}{2}M^2\right)}=\frac{\left(1-M^2\right)}{\left[1+\frac{(\gamma-1)}{2}M^2\right]}$$

$$\frac{dM^2}{M^2}\left[\frac{1-M^2}{1+\left(\frac{\gamma-1}{2}\right)M^2}\right]=M^2\gamma\left[\frac{dT_0}{T_0}+\frac{4c_f\,dx}{D}\right]+\frac{dT_0}{T_0}$$

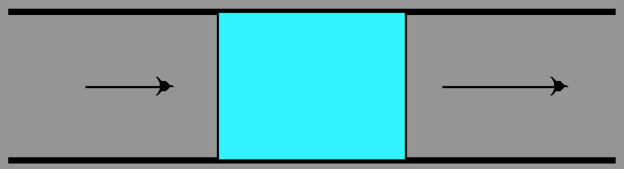
$$=\left[\left(1+\gamma M^2\right)\frac{dT_0}{T_0}+\frac{\gamma M^2 4c_f\,dx}{D}\right]$$

$$\frac{dM^2}{M^2}=\frac{\left[1+\frac{(\gamma-1)}{2}M^2\right]}{\left(1-M^2\right)}\left\{\left(1+\gamma M^2\right)\frac{dT_0}{T_0}+\frac{\gamma M^2 4c_f\,dx}{D}\right\}$$

or

$$\frac{dM^2}{dx}=\frac{M^2}{\left(1-M^2\right)}\left[1+\frac{(\gamma-1)}{2}M^2\right]\left\{\frac{\left(1+\gamma M^2\right)}{T_0}\frac{dT_0}{dx}+\frac{\gamma M^2 4c_f}{D}\right\}$$

Flow with heat addition and friction



The final form of the equation is as follows (see handout for details).

$$\frac{dM^2}{dx} = \frac{M^2\left(1+\frac{(\gamma-1)}{2}M^2\right)}{1-M^2}\left\{\frac{\left(1+\gamma M^2\right)}{T_0}\frac{dT_0}{dx}+\frac{\gamma M^2 4c_f}{D}\right\}$$

Flow with heat addition and friction

The above ODE can be integrated by using methods such as Runge-Kutta or using software packages such as Matlab which has routines for solving ODEs