AE/ME 339 Computational Fluid Dynamics (CFD)

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Inviscid Burger's Equation

It is a model equation used to test finite difference techniques
Inviscid and viscous forms can be used
Has a time dependent term, non-linear term similar to the convection term, and a viscous dissipation term

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{time-dependent term}} + \underbrace{u\frac{\partial u}{\partial x}}_{\text{convection term}} = \underbrace{\mu\frac{\partial^2 u}{\partial x^2}}_{\text{viscous dissipation term}} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Equation (1) is parabolic when the viscous dissipation term is included. With only the terms on the LHS, the equation is hyperbolic.

Inviscid form

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{time-dependent term}} + \underbrace{u\frac{\partial u}{\partial x}}_{\text{convection term}} = 0 \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Equation 2 can be thought of as the non-linear wave equation, where each point on the wave can propagate with a different speed.

Linear wave equation

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{time-dependent term}} + \underbrace{a\frac{\partial u}{\partial x}}_{\text{convection term}} = 0 \quad \text{.............(3)}$$

where a is the constant wave propagation speed.

As a result of u being a variable in Equation (2), the wave can distort and form discontinuities (Liepmann and Roshko,1957).

Let us consider the following equation

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0 \qquad (3)$$

where $F = F(u)$

Rewrite as $\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \qquad (4)$

where $A = \frac{\partial F}{\partial u} \qquad (5)$

Equation (3) could represent a vector

in which case $A = \frac{\partial F_i}{\partial u_j}$ is the Jacobian matrix

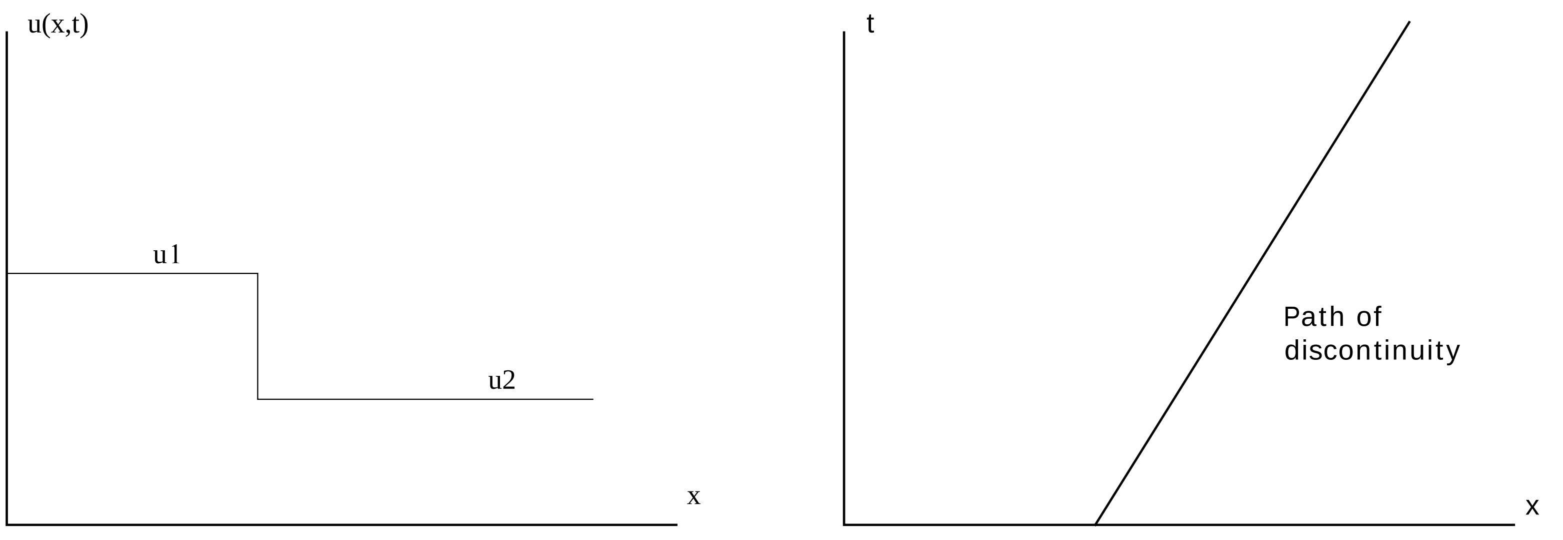

Figure 1. Traveling discontinuity proble for Burger's equation

It can be shown that the discontinuity travels with the speed (Tannehill et al., 1997)

$$u = \frac{dx}{dt} = \frac{u_1 + u_2}{2} \qquad (6)$$

See Figure 1.

Consider the initial data u(x,0) shown in Figure 2.
The characteristic for Burger's equation is given by

$$\frac{dt}{dx} = \frac{1}{u} \qquad (7)$$

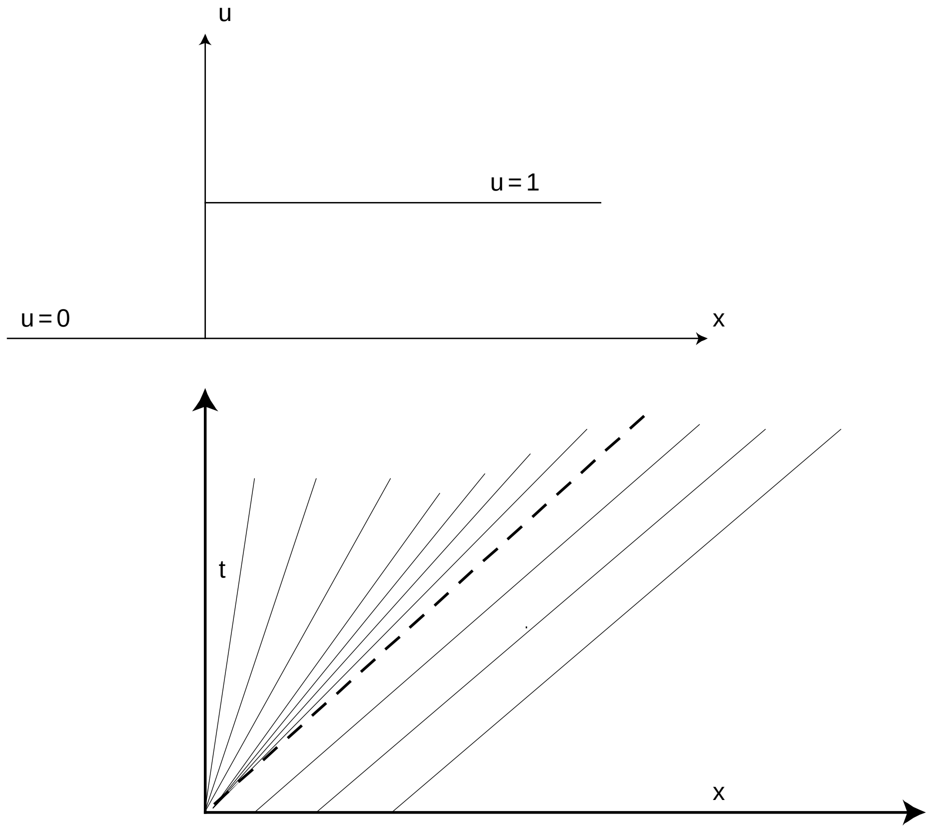

Figure 2. Characteristics for Centered Expansion

Figure 2 shows the characteristic diagram plotted in the (x, t) space.

Bounded by the x = 0 (vertical) line and the characteristic denoted by the dashed line.

Solution can be written as

$$u = 0 \qquad x \leq 0$$

$$u = \frac{x}{t} \qquad 0 < x < t$$

$$u = 1 \qquad x \geq t$$

The initial distribution of u results in a centered expansion where the width of the expansion grows linearly with time.

The above solutions can now be used to evaluate finite difference algorithms.

Implicit methods

Time-centered implicit method (Beam-Warming, 1976)

Consider the the following two Taylor series expansions

$$u_j^{n+1} = u_j^n + \Delta t (u_t)_j^n + \frac{(\Delta t)^2}{2}(u_{tt})_j^n + \frac{(\Delta t)^3}{6}(u_{ttt})_j^n + \ldots \qquad (8)$$

$$u_j^n = u_j^{n+1} - \Delta t (u_t)_j^{n+1} + \frac{(\Delta t)^2}{2}(u_{tt})_j^{n+1} - \frac{(\Delta t)^3}{6}(u_{ttt})_j^{n+1} + \ldots \qquad (9)$$

Subtract Equation (9) from Equation (8)

$$u_j^{n+1} - u_j^n = u_j^n - u_j^{n+1} + \Delta t \left\{ (u_t)_j^n + (u_t)_j^{n+1} \right\} +$$

$$\frac{(\Delta t)^2}{2}\left\{ (u_{tt})_j^n - (u_{tt})_j^{n+1} \right\} + O[(\Delta t)^3] \qquad (10)$$

$(u_{tt})_j^{n+1}$ can be substituted in Eq. (10) using the following Taylor series expansion

$$(u_{tt})_j^{n+1} = (u_{tt})_j^n + \Delta t (u_{ttt})_j^n + \ldots \qquad (11)$$

$$2u_j^{n+1} = 2u_j^n + \Delta t \left\{ (u_t)_j^n + (u_t)_j^{n+1} \right\} +$$

$$\frac{(\Delta t)^2}{2} \left\{ (u_{tt})_j^n - (u_{tt})_j^n - \Delta t (u_{ttt})_j^n \right\} + O[(\Delta t)^3] \qquad (12)$$

Which reduces to

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{2} \left\{ (u_t)_j^n + (u_t)_j^{n+1} \right\} + O[(\Delta t)^3] \qquad (13)$$

Now we substitute the wave equation $u_t = -au_x$ to get the following

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2}\left\{(u_x)_j^n + (u_x)_j^{n+1}\right\} + O[(\Delta t)^3] \qquad (14)$$

and now replace the u_x terms by 2nd order central differences

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{4\Delta x}\left\{(u)_{j+1}^n - (u)_{j-1}^n + (u)_{j+1}^{n+1} - (u)_{j-1}^{n+1}\right\} + O[(\Delta t)^3] \qquad (15)$$

The method is 2nd order accurate ($\varepsilon = O[(\Delta t)^2,(\Delta x)^2]$) and unconditionally stable for all time steps.
A tridiagonal system must be solved for each time step

The Beam-Warming method can now be applied to the inviscid Burger's equation

Substituting in Eq. (14) using Eq. (3) gives

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2}\left\{\left(\frac{\partial F}{\partial x}\right)^n + \left(\frac{\partial F}{\partial x}\right)^{n+1}\right\} \quad (15)$$

The above is a non-linear problem since $F = F(u)$.

Linearization or iteration is therefore necessary

Beam and Warming (1976) suggested the following

$$F^{n+1} \approx F^n + \left(\frac{\partial F}{\partial u}\right)^n (u^{n+1} - u^n) = F^n + A^n (u^{n+1} - u^n) \quad (16)$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{2}\left\{2\left(\frac{\partial F}{\partial x}\right)^n + \frac{\partial}{\partial x}\left[A^n (u^{n+1} - u^n)\right]\right\} \quad (17)$$

Replacing the x-derivatives using 2nd order CD would yield the following

$$-\frac{1}{4}\frac{\Delta t}{\Delta x}A_{j-1}^{n}u_{j-1}^{n+1}+u_{j}^{n+1}+\frac{1}{4}\frac{\Delta t}{\Delta x}A_{j+1}^{n}u_{j+1}^{n+1}=$$

$$-\frac{\Delta t}{\Delta x}\frac{F_{j+1}^{n}-F_{j-1}^{n}}{2}-\frac{1}{4}\frac{\Delta t}{\Delta x}A_{j-1}^{n}u_{j-1}^{n}+u_{j}^{n}+\frac{1}{4}\frac{\Delta t}{\Delta x}A_{j+1}^{n}u_{j+1}^{n} \quad (18)$$

The Jacobian A has a single element for the Burger's equation.

Eq. (18) represents linear tridiagonal system.

Solution by Thomas algorithm is feasible.

Beam and Warming suggests the following explicit artificial viscosity term

$$D = -\frac{\omega}{8}\left(u_{j+2}^{n} + u_{j+1}^{n} + u_{j}^{n} + u_{j-1}^{n} + u_{j-2}^{n}\right) \quad (19)$$

Recommended values of ω lie in the range

$$0 \le \omega \le 1$$

Delta Form

Some times it is better to write the equation for change in the variable from time level n to (n+1). Eq. (18) then becomes

$$-\frac{1}{4}\frac{\Delta t}{\Delta x}A_{j-1}^{n}\Delta u_{j-1}^{n+1}+\Delta u_{j}^{n+1}+\frac{1}{4}\frac{\Delta t}{\Delta x}A_{j+1}^{n}\Delta u_{j+1}^{n+1}=$$

$$-\frac{\Delta t}{\Delta x}\frac{F_{j+1}^{n}-F_{j-1}^{n}}{2} \qquad (20)$$

The delta form reduces the number of arithmatic operations since the RHS has only one term. Also round-off error will be smaller in this case.

Some Examples

Solution of Burger's equation

Use MacCormack's method to solve inviscid Burger's equation using a mesh with 51 points in the x-direction. Solve the equation for a right propagating discontinuity with u = 1 at the first 11 nodes and u = 0 at the rest of the nodes.
Use Courant number = 1.

Solution

MacCormack's method

$$\bar{u}_j = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1}^n - F_j^n\right)$$

$$u_j^{n+1} = \frac{1}{2}\left[u_j^n + \bar{u}_j - \frac{\Delta t}{\Delta x}\left(\bar{F}_j - \bar{F}_{j-1}\right)\right]$$