AE/ME 339 Computational Fluid Dynamics (CFD)

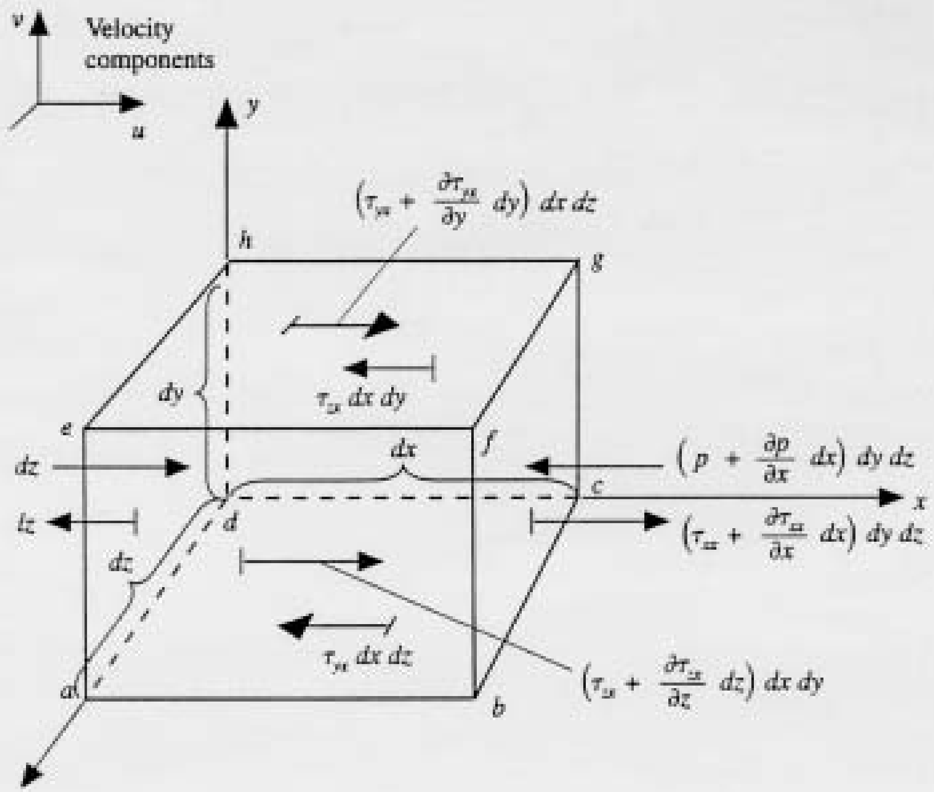
K. M. Isaac
Professor of Aerospace Engineering

“........in the phrase ‘computational fluid dynamics’ the word ‘computational’ is simply an adjective to ‘fluid dynamics.’..........”

-John D. Anderson

K. M. Isaac
MAEEM Dept., UMR

Momentum equation



.8
simally small, moving fluid element. Only the forces in the x direction are shown. Model used derivation of the x component of the momentum equation.

Consider the moving fluid element model shown in Figure 2.2b

Basis is Newton's 2nd Law which says **F** = m **a**

Note that this is a vector equation.

It can be written in terms of the three cartesian scalar components, the first of which becomes

$$F_x = m\, a_x$$

Since we are considering a fluid element moving with the fluid, its mass, *m*, is fixed.

The momentum equation will be obtained by writing expressions for the externally applied force, F_x, on the fluid element and the acceleration, a_x, of the fluid element.

The externally applied forces can be divided into two types:

1. Body forces: Distributed throughout the control volume. Therefore, this is proportional to the volume. Examples: gravitational forces, magnetic forces, electrostatic forces.

2. Surface forces: Distributed at the control volume surface. Proportional to the surface area. Examples: forces due to surface and normal stresses. These can be calculated from stress-strain rate relations.

Body force on the fluid element = $f_x \rho .(dx\ dy\ dz)$
where fx is the body force per unit mass in the x-direction

The shear and normal stresses arise from the deformation of the fluid element as it flow along. The shape as well as the volume of the fluid element could change and the associated normal and tangential stresses give rise to the surface stresses.

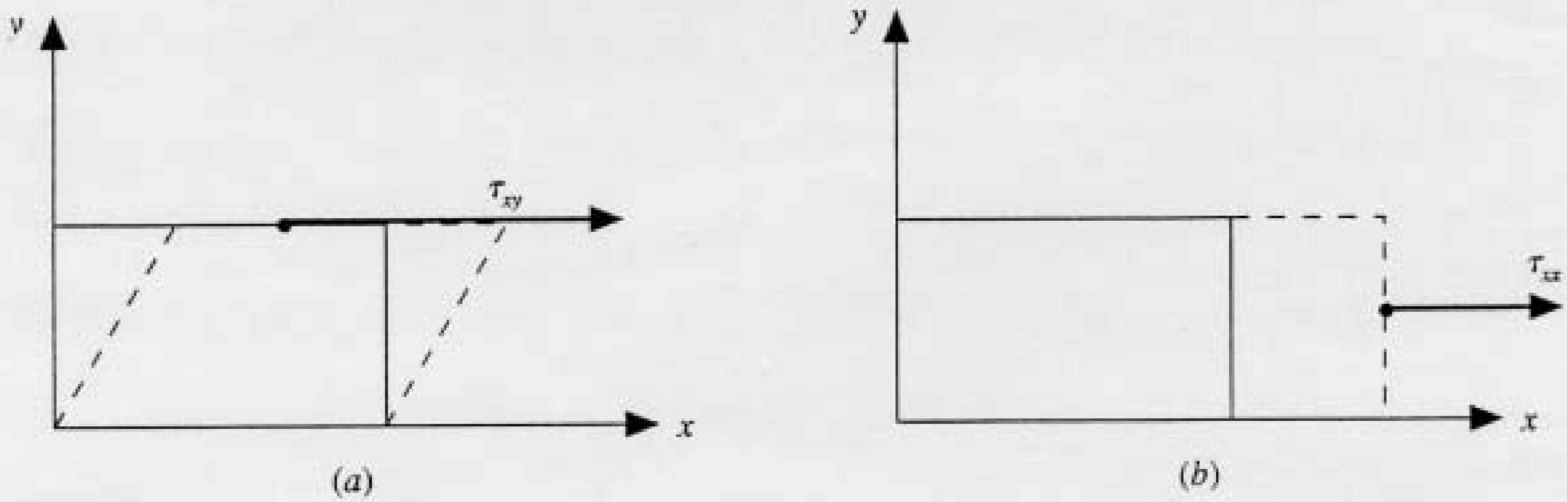


FIG. 2.9
Illustration of (*a*) shear stress (related to the time rate of charge of the shearing deformation and (*b*) normal stress (related to the time rate of charge of volume).

The relation between stress and rate of strain in a fluid is known from the type of fluid we are dealing with.
Most of our discussion will relate to Newtonian fluids for which

Stress is proportional to the rate of strain

For non-Newtonian fluids more complex relationships should be used.

Notation: stress τ_{ij} indicates stress acting on a plane perpendicular to the *i*-direction (x-axis) and the stress acts in the direction, *j*, (y-axis).

The stresses on the various faces of the fluid element can written as shown in Figure 2.8. Note the use of Taylor series to write the stress components.

The normal stresses also has the pressure term.

Net surface force acting in x direction =

$$\left[p-\left(p+\frac{\partial p}{\partial x}dx\right)\right]dydz$$

$$+\left[\left(\tau_{xx}+\frac{\partial \tau_{xx}}{\partial x}dx\right)-\tau_{xx}\right]dydz+\left[\left(\tau_{yx}+\frac{\partial \tau_{yx}}{\partial y}dy\right)-\tau_{yx}\right]dxdz$$

$$+\left[\left(\tau_{zx}+\frac{\partial \tau_{zx}}{\partial z}dz\right)-\tau_{zx}\right]dxdy\text{............(2.46)}$$

$$F_x = \left[-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right] dxdydz + \rho f_x dxdydz ....(2.47)$$

$$m = \rho dxdydz ....(2.48)$$

$$a_x = \frac{Du}{Dt} ....(2.49)$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x \text{.........}(2.50a)$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y \text{.........}(2.50b)$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z \text{.........}(2.50c)$$

$$\rho \frac{Du}{Dt} = \rho \frac{\partial u}{\partial t} + \rho \overline{V} \cdot \overline{\nabla} u \text{.....................} (2.51)$$

$$\frac{\partial(\rho u)}{\partial t} = \rho \frac{\partial u}{\partial t} + u \frac{\partial \rho}{\partial t}$$

$$\rho \frac{\partial u}{\partial t} = \frac{\partial(\rho u)}{\partial t} - u \frac{\partial \rho}{\partial t} \text{.............} (2.52)$$

$$\rho \overline{V} \cdot \overline{\nabla} u = \nabla \cdot \left(\rho u \overline{V}\right) - u \overline{\nabla} \cdot \left(\rho \overline{V}\right) \text{........} (2.53)$$

$$\rho \frac{Du}{Dt} = \frac{\partial(\rho u)}{\partial t} - u \frac{\partial \rho}{\partial t} - u \overline{\nabla} \cdot \left(\rho \overline{V}\right) + \overline{\nabla} \cdot \left(\rho u \overline{V}\right)$$

$$= \frac{\partial(\rho u)}{\partial t} - u \left[\frac{\partial \rho}{\partial t} + \overline{\nabla} \cdot \left(\rho \overline{V}\right)\right] + \overline{\nabla} \cdot \left(\rho u \overline{V}\right) \text{.............} (2.54)$$

The term in the brackets is zero (continuity equation)
The above equation simplifies to

$$\rho \frac{Du}{Dt} = \frac{\partial(\rho u)}{\partial t} + \bar{\nabla} \cdot \left(\rho u \bar{V}\right) \text{...........(2.55)}$$

Substitute Eq. (2.55) into Eq. (2.50a) shows how the following equations can be obtained.

$$\frac{\partial(\rho u)}{\partial t} + \bar{\nabla} \cdot \left(\rho u \bar{V}\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x \text{..........}(2.56a)$$

$$\frac{\partial(\rho v)}{\partial t} + \bar{\nabla} \cdot \left(\rho v \bar{V}\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y \text{..........}(2.56b)$$

$$\frac{\partial(\rho w)}{\partial t} + \bar{\nabla} \cdot \left(\rho w \bar{V}\right) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho f_z \text{..........}(2.56c)$$

The above are the Navier-Stokes equations in “conservation form.”

For Newtonian fluids the stresses can be expressed as follows

$$\tau_{xx} = \lambda\left(\bar{\nabla}\cdot\bar{V}\right) + 2\mu\frac{\partial u}{\partial x} \quad \text{...........}(2.57a)$$

$$\tau_{yy} = \lambda\left(\bar{\nabla}\cdot\bar{V}\right) + 2\mu\frac{\partial v}{\partial y} \quad \text{...........}(2.57b)$$

$$\tau_{zz} = \lambda\left(\bar{\nabla}\cdot\bar{V}\right) + 2\mu\frac{\partial w}{\partial z} \quad \text{...........}(2.57c)$$

$$\tau_{xy} = \tau_{yx} = \mu\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] \quad \text{...........}(2.57d)$$

$$\tau_{xz} = \tau_{zx} = \mu\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \quad \text{...........}(2.57e)$$

$$\tau_{yz} = \tau_{zy} = \mu\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right] \quad \text{...........}(2.57f)$$

In the above μ is the coefficient of dynamic viscosity and λ is the second viscosity coefficient.
Stokes hypothesis given below can be used to relate the above two coefficients

$$\lambda = -2/3\ \mu$$

The above can be used to get the Navier-Stokes equations in the following familiar form

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u^2)}{\partial x}+\frac{\partial(\rho uv)}{\partial y}+\frac{\partial(\rho uw)}{\partial z}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(\lambda\nabla * V+2\mu\frac{\partial u}{\partial x}\right)$$

$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\rho f_x ....(2.58a)$$

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho v^2)}{\partial y}+\frac{\partial(\rho uv)}{\partial x}+\frac{\partial(\rho vw)}{\partial z}=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\left(\lambda\nabla * V+2\mu\frac{\partial v}{\partial y}\right)$$

$$+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\rho f_y ....(2.58b)$$

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial\left(\rho w^2\right)}{\partial z}+\frac{\partial(\rho uw)}{\partial x}+\frac{\partial(\rho vw)}{\partial y}=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left(\lambda\nabla * V+2\mu\frac{\partial w}{\partial z}\right)$$

$$+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\rho f_z ....(2.58c)$$

The energy equation can also be derived in a similar manner.
Read Section 2.7

For a summary of the equations in conservation and non-conservation forms see Anderson, pages 76 and 77.

The above equations can be simplified for inviscid flows by dropping the terms involving viscosity. (read Section 2.8.2)