UNSTEADY-STATE HEAT CONDUCTION IN A LONG BAR OF SQUARE CROSS SECTION (IMPLICIT ALTERNATING-DIRECTION METHOD)

Problem Statement

An infinitely long bar of thermal diffusivity α has a square cross section of side $2a$. It is initially at a uniform temperature θ_0 and then suddenly has its surface maintained at a temperature θ_1. Compute the subsequent temperatures $\theta(x,y,t)$ inside the bar.

Method of Solution

If dimensionless distances, time, and temperature are defined by

$$X = \frac{x}{a}, \quad Y = \frac{y}{a}, \quad \tau = \frac{\alpha t}{a^2}, \quad \text{and} \quad T = \frac{\theta - \theta_0}{\theta_1 - \theta_0},$$

it may be shown that the unsteady-state conduction is governed by

$$\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} = \frac{\partial T}{\partial \tau}. \tag{7.3.1}$$

Because of symmetry, it suffices to solve the problem in one quadrant only, such as that shown in Fig. 7.3.1. The center of the bar ($X = 0$, $Y = 0$) and one of its corners ($X = 1$, $Y = 1$) are regarded as the grid points (0,0) and (n,n), respectively. From symmetry, there is no heat flux across the X and Y axes, which behave, in effect, as perfectly insulating boundaries across which the normal temperature gradient is zero. The initial and boundary conditions are:

$\tau = 0$: $T = 0$ throughout the region,
$\tau > 0$: $T = 1$ along the sides $X = 1$ and $Y = 1$,
$\partial T/\partial X = 0$ and $\partial T/\partial Y = 0$ along the sides $X = 0$ and $Y = 0$, respectively.

The solution to the problem is by the implicit alternating-direction method described in the text and summarized by equations (7.53a) and (7.53b), with the first half time-step implicit in the X direction. Let T and T^* refer to temperatures at the beginning and end of a half

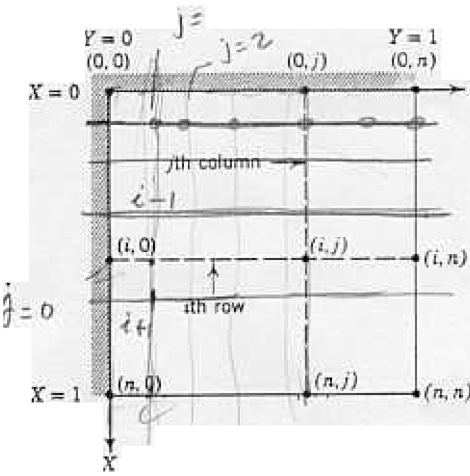

Figure 7.3.1 *Lower right-hand quadrant of cross section of bar.*

time-step $\Delta\tau/2$. Equation (7.53a) is applied to each point $i = 1, 2, \ldots, n-1$ in the jth column; also, the method of Section 7.17 is used in conjunction with the effective boundary condition $\partial T/\partial X = 0$ at $X = 0$ to yield a finite-difference approximation of equation (7.3.1) at the boundary point $(0, j)$. We then have the following tridiagonal system for the jth column:

$$\left.\begin{array}{l}
bT^*_{0,j} - 2T^*_{1,j} = d_0 \\
-T^*_{0,j} + bT^*_{1,j} - T^*_{2,j} = d_1 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-T^*_{i-1,j} + bT^*_{i,j} - T^*_{i+1,j} = d_i \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-T^*_{n-3,j} + bT^*_{n-2,j} - T^*_{n-1,j} = d_{n-2} \\
-T^*_{n-2,j} + bT^*_{n-1,j} = d_{n-1}
\end{array}\right\} \tag{7.3.2}$$

$$\left.\begin{array}{ll}
d_i = T_{i,j-1} + fT_{i,j} + T_{i,j+1}, & \text{for} \quad i = 0, 1, \ldots, n-2 \\
d_{n-1} = T_{n-1,j-1} + fT_{n-1,j} + T_{n-1,j+1} + T_{n,j} &
\end{array}\right\} \text{for } j \neq 0,$$

$$\left.\begin{array}{ll}
d_i = 2T_{i,1} + fT_{i,0}, & \text{for} \quad i = 0, 1, \ldots, n-2 \\
d_{n-1} = 2T_{n-1,1} + fT_{n-1,0} + T_{n,0} &
\end{array}\right\} \text{for } j = 0,$$

bottom BC.

where

$$b = 2(1/\lambda + 1),$$
$$f = 2(1/\lambda - 1),$$
$$\lambda = \Delta\tau/(\Delta x)^2.$$

Flow Diagram