Computational Fluid Dynamics (AE/ME 339)
MAE Dept.

Home Work Problem

Transform the simplified equations of free convection flow into non-dimensional form using the following non-dimensionalization scheme. Assume $\alpha = \nu$.

$$\frac{\partial u}{\partial x} + \frac{\partial \text{v}}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \text{v}\frac{\partial u}{\partial y} = g_x\beta(T - T_\infty) + \nu\frac{\partial^2 u}{\partial y^2}$$

where β is the coefficient of thermal expansion.

$$\beta = -\frac{1}{\rho}\frac{d\rho}{dT}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + \text{v}\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

$$\xi = x(g\beta\Delta T/\nu^2)^{\frac{1}{3}}$$

$$\eta = y(g\beta\Delta T/\nu^2)^{\frac{1}{3}}$$

$$\tau = t(g\beta\Delta T)^{\frac{2}{3}}/\nu^{\frac{1}{3}}$$

$$\bar{u} = u/(\nu g\beta\Delta T)^{\frac{1}{3}}$$

$$\bar{\text{v}} = \text{v}/(\nu g\beta\Delta T)^{\frac{1}{3}}$$

$$\theta = \frac{T - T_\infty}{T_1 - T_\infty} = \frac{T - T_\infty}{\Delta T}$$

Equations in non-dimensional Form

$$\text{continuity:} \quad \frac{\partial \overline{u}}{\partial \xi} + \frac{\partial \overline{\mathrm{v}}}{\partial \eta} = 0$$

$$\text{u-momentum:} \quad \frac{\partial \overline{u}}{\partial \tau} + \overline{u}\frac{\partial \overline{u}}{\partial \xi} + \overline{\mathrm{v}}\frac{\partial \overline{u}}{\partial \eta} = \theta + \frac{\partial^2 \overline{u}}{\partial \eta^2} \qquad (1)$$

$$\text{energy:} \quad \frac{\partial \theta}{\partial \tau} + \overline{u}\frac{\partial \theta}{\partial \xi} + \overline{\mathrm{v}}\frac{\partial \theta}{\partial \eta} = \frac{1}{\mathrm{Pr}}\frac{\partial^2 \theta}{\partial \eta^2}$$

Initial and boundary conditions:

$$\begin{aligned}
&\tau = 0: \quad \overline{\mathrm{u}} = 0, \overline{\mathrm{v}} = 0, \theta = 0 \\
&\tau > 0: \\
&\xi = 0: \quad \overline{\mathrm{u}} = 0, \overline{\mathrm{v}} = 0, \theta = 0 \\
&\eta = 0: \quad \overline{\mathrm{u}} = 0, \overline{\mathrm{v}} = 0, \theta = 1 \\
&\eta = \infty: \quad \overline{\mathrm{u}} = 0, \theta = 0
\end{aligned} \qquad (2)$$

Finite difference formulation

$$\frac{\theta_{i,j}^{n+1} - \theta_{i,j}^{n}}{\Delta\tau} + \overline{u}_{i,j}^{n}\frac{\theta_{i,j}^{n} - \theta_{i-1,j}^{n}}{\Delta\xi} + \overline{\mathrm{v}}_{i,j}^{n}\frac{\theta_{i,j+1}^{n} - \theta_{i,j}^{n}}{\Delta\eta} = \frac{1}{\mathrm{Pr}}\frac{\theta_{i,j-1}^{n} - 2\theta_{i,j}^{n} + \theta_{i,j+1}^{n}}{\left(\Delta\eta\right)^2}$$

$$\frac{\overline{u}_{i,j}^{n+1} - \overline{u}_{i,j}^{n}}{\Delta\tau} + \overline{u}_{i,j}^{n}\frac{\overline{u}_{i,j}^{n} - \overline{u}_{i-1,j}^{n}}{\Delta\xi} + \overline{\mathrm{v}}_{i,j}^{n}\frac{\overline{u}_{i,j+1}^{n} - \overline{u}_{i,j}^{n}}{\Delta\eta} = \theta_{i,j}^{n} + \frac{\overline{u}_{i,j-1}^{n} - 2\overline{u}_{i,j}^{n} + \overline{u}_{i,j+1}^{n}}{\left(\Delta\eta\right)^2} \qquad (3)$$

$$\frac{\overline{u}_{i,j}^{n+1} - \overline{u}_{i-1,j}^{n+1}}{\Delta\xi} + \frac{\overline{\mathrm{v}}_{i,j}^{n+1} - \overline{\mathrm{v}}_{i,j-1}^{n+1}}{\Delta\eta} = 0$$

Note the special features of the formulation. It is explicit, solved in the order: energy, momentum and continuity. Both forward and backward difference forms are used for the first derivative in the energy and momentum equations. The continuity equation is solved by using the updated values of the x-component of velocity.