ME/AE 339

Computational Fluid Dynamics

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Topic1_ODE

Course Outline

- Ordinary differential equations (ODE)
- Numerical techniques for solving ODEs
- Example: Laminar Free Jet
- Partial differential equations, classification
- Discretization of derivatives
- Errors and analysis of stability
- Example: Unsteady heat conduction in a rod
- Example: Natural convection at a heated vertical plate
- Discretization techniques

Course Outline (continued)

- Couette flow
- The shock tube problem
- Introduction to packaged codes:
  - Grid generation
  - Problem setup
  - Solution
- Turbulence modeling

ODEs and PDEs may be discretized-approximated-
as a set of algebraic equations and solved

Discretization methods for ODEs are well known

e.g., Runge-Kutta methods for initial value problems
and shooting methods for BV problems

PDEs involve more than 1 independent variable
e.g., x, y, z, t in Cartesian coordinates for time-dependent
Problems

PDEs can be discretized using finite difference
Methods

PDEs can also be discretized in integral form, known as finite volume methods

Sometimes coordinate transformation is necessary before discretization

Plane Laminar Free Jet

Ref: F. M. White, *Viscous Fluid Flow*
The jet emerges from a slit into ambient quiescent air at x = 0
The static pressure can be assumed to be constant.
Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}=0$$

Momentum equation

$$u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}\approx \nu\frac{\partial^2 u}{\partial y^2}$$

The momentum flux is constant in the x-direction, i.e.,

$$J=\int_{-\infty}^{+\infty}u^2dy=const$$

Stream function

$$\psi=\nu^{1/2}x^{1/3}f(\eta)$$

where

$$\eta=\frac{y}{3\nu^{1/2}x^{2/3}}$$

Velocity components

$$u=\frac{f'(\eta)}{3x^{1/3}},\quad \mathrm{v}=\frac{\nu^{1/2}}{3x^{2/3}}\left[f-2f'(\eta)\right]$$

Where ’ denotes differentiation with respect to η.
The continuity and momentum equations can be combined a written in terms of the new variables as

$$f'''+ff''+f'^2=0$$

The above is a third order ordinary differential equation

Solution

Boundary conditions:
1. Symmetry along x-axis

$v = 0$ at $y = 0$

$\frac{\partial u}{\partial y} = 0$ at $y = 0$

2. Quiescent ambient fluid

$u = 0$ at $y = \infty$

The above boundary conditions can be written in terms of f as

$f(0) = 0, \quad f''(0) = 0, \quad \text{and} \quad f'(\infty) = 0$

This is known as a boundary value problem.
Boundary conditions are specified at two boundaries (as opposed to an initial value problem in which all the boundary conditions are specified at one location.
IV Solution methods such as RK can be used as follows:
Guess f'(0) and solve. Check if $f'(\infty) = 0$
If not adjust the value of f'(0) so as to make $f'(\infty) = 0$ to within a specified tolerance.

Runge-Kutta Method

4th order method:

Let the first order ODE be represented as

$$\frac{dy}{dx} = f(x, y)$$

The 4th order RK-Gill algorithm is then given by

$$y_{i+1} = y_i + \frac{h}{6}\left[k_1 + 2\left(1 - \frac{1}{\sqrt{2}}\right)k_2 + 2\left(1 + \frac{1}{\sqrt{2}}\right)k_3 + k_4\right]$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_1\right)$$

$$k_3 = f\left[x_i + \frac{1}{2}h, y_i + \left(-\frac{1}{2} + \frac{1}{\sqrt{2}}\right)hk_1 + \left(1 - \frac{1}{\sqrt{2}}\right)hk_2\right]$$

$$k_4 = f\left[x_i + h, y_i - \frac{1}{\sqrt{2}}hk_2 + \left(1 + \frac{1}{\sqrt{2}}\right)hk_3\right]$$

The method can be used for several simultaneous first-order equations as well as a single higher-order equation.
See any numerical methods book for details.

Program Completed

University of Missouri-Rolla