



ME/AE 339

Computational Fluid Dynamics

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Computational Fluid Dynamics (AE/ME 339)
PDE

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Partial Differential Equations (PDE) (CLW: 7.1, 7.3, 7.4)

PDE's can be linear or nonlinear

Order : Determined by the order of the highest derivative.

Linear, 2nd order PDE's are classified as the elliptic, hyperbolic
Parabolic type.

Example:

$$A_1 \frac{\partial^2 u}{\partial x^2} + A_2 \frac{\partial^2 u}{\partial y^2} + A_3 \frac{\partial^2 u}{\partial z^2} + B_1 \frac{\partial u}{\partial x} \\ + B_2 \frac{\partial u}{\partial y} + B_3 \frac{\partial u}{\partial z} + C u + D = 0$$

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Coefficients A_1, A_2, A_3 May be +1, -1 or zero.

u is the dependent variable and x, y, z are the independent variables.
Note that we do not have any cross-derivative terms.

Classification:

Elliptic: A_1, A_2, A_3 are non-zero and have the same
sign, then PDE is of the elliptic type.

Hyperbolic:

A_1, A_2, A_3 are non-zero and of mixed sign, the PDE is
hyperbolic

Parabolic:

If one of A_1, A_2, A_3 , say A_2 is zero and the rest are of the same
sign, and if B_2 is non-zero, the PDE is parabolic.

Since A_i, B_i, C and D may be functions of x, y and z , the
classification may depend on position in space.

In many CFD problems, one of the independent variables will
be time and the rest will be space coordinates such as x, y, z or
or transformed variables such as ξ, η, ζ .

Two-Dimensional Examples

Elliptic

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hyperbolic

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

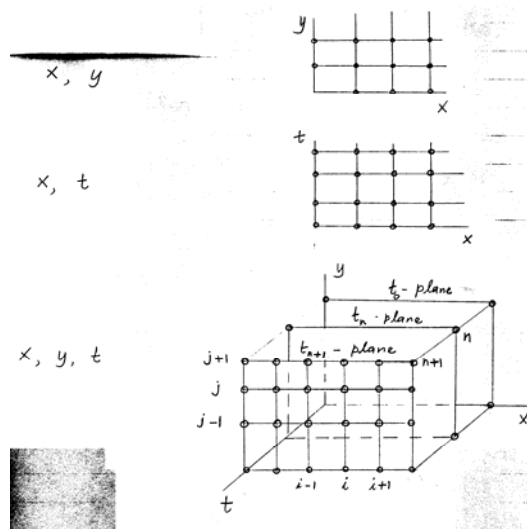
Parabolic

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Numerical solution of PDE requires a finite number of points to
Discretize the equations.

Examples:

See Figure in the next slide.



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Solution requires initial and boundary conditions depending on the Problem.

Indices i, j, n can be used to label the nodes in x, y, t directions(see fig.)

If the origin has $i=0, j=0$ and $n = 0$, then the node i, j, n has coordinates $i\Delta x, j\Delta y, n\Delta t$,

where $\Delta x, \Delta y, \Delta t$ are the uniform intervals

between nodes along x, y, t coordinate directions.

Let $u(x, y, t) \equiv u_{i,j,n}$ be the exact solution of the PDE and $V_{i,j,n}$

be the approximations to be determined at each grid point.

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The derivatives of the original PDE are approximated using the symbol

$V_{i,j,n}$ and the discretization intervals $\Delta x, \Delta y, \Delta t$.

The procedure leads to a set of algebraic equations of $V_{i,j,n}$ which are then solved.

Fine grids can be used to obtain solutions $V_{i,j,n}$ that are close to $U_{i,j,n}$.

Examples of PDE's common in engineering

1. Unsteady heat conduction equation

1D form:
$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) = c_p \rho \frac{\partial T}{\partial t}$$

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T - temperature
k - thermal conductivity
 ρ - density
 c_p - specific heat

If k is a constant, the equation becomes

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

Where $\alpha \equiv \frac{k}{c_p \rho}$ is the thermal diffusivity.

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In CFD , normalization of variables are often used to improve the solution (for proper scaling of the variables).

$$\text{Let } \xi = \frac{x}{L}, \tau = \frac{\alpha t}{L^2}$$

for heat conduction in a rod of length L.

Figure.

The PDE then becomes

$$\frac{\partial^2 T}{\partial \xi^2} = \frac{\partial T}{\partial \tau}$$

It is also possible to non-dimensionalize the dependent variable T.

Taylor's Expansion.

$$\text{Let } \frac{dy}{dx} = f(x, y). \quad \text{Taylor series is as follows.} \quad (1)$$

$$y(x_0 + h) = y(x_0) + hf(x_0, y(x_0)) + \frac{h^2}{2!} f'(x_0, y(x_0)) + \frac{h^3}{3!} f''(x_0, y(x_0)) + \dots \quad (2)$$

$$\text{Where } f'(x, y(x)) = \frac{df}{dx}(x, y(x))$$

$$f''(x, y(x)) = \frac{d^2 f}{dx^2}(x, y(x))$$

The higher order derivatives in Eq.(2) can be determined by Differentiating Eq.(1) by chain rule. i. e.,

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

Example 1 : f(x,y) is a function of x alone.

$$\frac{dy}{dx} = x^2$$

$$f'(x, y) = 2x$$

$$f'(x_0, y_0) = 2x_0$$

$$f''(x, y) = 2$$

$$f''(x_0, y_0) = 2$$

$$f'''(x, y) = 0$$

$$f'''(x_0, y_0) = 0$$

.....

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.....

.....

$$f^n(x, y) = 0$$

$$f^n(x_0, y_0) = 0$$

The function at the neighboring point ($x = x_0+h$) becomes

$$y(x_0 + h) = y(x_0) + hx_0^2 + h^2x_0 + \frac{h^3}{3}$$

Example 2: $f(x,y)$ is a function of y alone.

$$\frac{dy}{dx} = f(x, y) = 2y$$

$$f'(x, y) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0 + 2 \times 2y = 4y$$

Similarly

$$f''(x, y) = 8y$$

$$f'''(x, y) = 16y$$

.....

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$$f^n(x, y) = 2^{(n+1)}y$$

$$y(x_0 + h) = y(x_0) + 2hy(x_0) + 4\frac{h^2}{2!}y(x_0) + 8\frac{h^3}{3!}y(x_0)$$
$$= y(x_0) \left[1 + 2h + \frac{(2h)^2}{2!} + \frac{(2h)^3}{3!} + \dots \right]$$

Taylor series can also be used for non-linear higher order equations.

Example:

$$\frac{d^2 y}{dx^2} - \frac{dy}{dx} + xy^2 = 0$$

$$\frac{dy}{dx} = f(x, y) = \frac{d^2 y}{dx^2} + xy^2$$

$$y'' = y' - xy^2$$

$$y''' = y'' - 2xyy' - y^2$$

$$y^{iv} = y''' - (2yy' + 2xyy'' + 2xy'^2) - 2yy'$$
$$y^{iv} = y''' - 4yy' - 2xy'^2 - 2xyy''$$

Consider the initial conditions

$$\text{At } x = 0, y(0) = 1, y'(0) = -1$$

$$y''(0) = y'(0) - 0 \times y^2(0) = -1$$
$$y'''(0) = y''(0) - y^2(0)$$
$$y'''(0) = -1 - 1 = -2$$
$$y^{iv}(0) = y'''(0) - 4 \times 1(-1)$$
$$= -2 + 4 = 2$$

Since Taylor series gives

$$y_{k+1} = y_k + y_k' \frac{h}{1!} + y_k'' \frac{h^2}{2!} + y_k''' \frac{h^3}{3!} + \dots$$

For $k = 0$, y_1 becomes

$$y_1 = 1 - h - \frac{h^2}{2!} - \frac{h^3}{3!} + \frac{h^4}{4!} + \dots$$

Letting $h=0.1$, we get

$$y_1 = 1 - 0.1 - \frac{0.01}{2} - \frac{0.001}{3} + \frac{0.0001}{12}$$

$$y_1 = 0.894675$$

Calculation of y_2

Need y_1'

Calculate y_1' by differentiating the expression for y_1
w.r.t. h .

$$y_1' = -1 - h - h^2 + \frac{h^3}{3} + \dots$$

$$y_1' = -1 - 0.1 - 0.01 + \frac{0.001}{3} = -1.1097$$

y_1'' , y_1''' , y_1^{iv} etc. can now be calculated as before and y_2
can be obtained.



Program Completed

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