AE/ME 339

Computational Fluid Dynamics (CFD)

K. M. Isaac

Implicit form of difference equation

In the previous explicit method, the solution at time level n, $u_{i,n}$, depended only on the known values of u, $u_{i-1,n-1}$, $u_{i,n-1}$, and $u_{i+1,n-1}$, all which are at time level n-1.

Note: $u_{i,n}$ and u_i^n mean the same.

We will be using these interchangeably.

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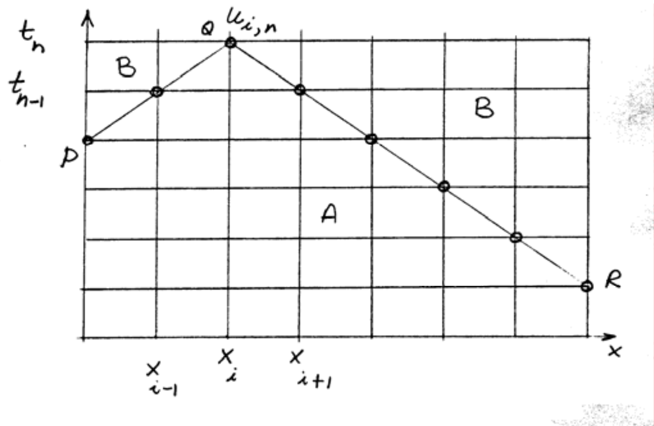
Nature of solution in explicit method can be illustrated graphically as shown below.



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In this formulation, solution at (i,n), $u_{i,n}$ is affected only by the values along and below boundary PQR (region A) in the previous figure.Values in region above PQR (region B) do not influence $u_{i,n}$.

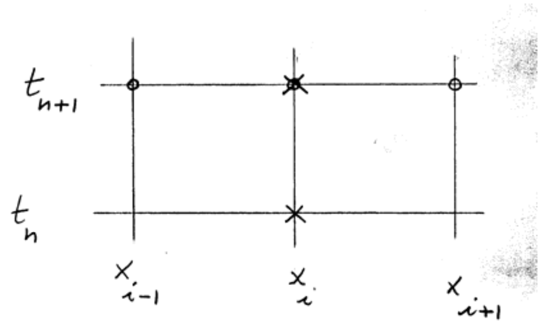
Exact solution u(x,y) at Q depends on the values at all times earlier than t_n , a property of parabolic PDE.

Is a limitation of the explicit method.

Stability criterion: $$0 \le \frac{\Delta t}{(\Delta x)^2} \le \frac{1}{2}$$

Fully Implicit Method

The nodes used in implicit method are illustrated in the figure below



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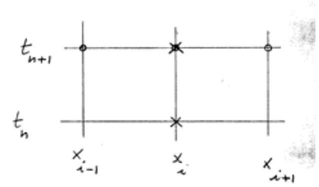
As before, crosses (x) denote grid points used for $\dfrac{\partial u}{\partial t}(u_t)$ and circles (o) for $\dfrac{\partial^2 u}{\partial x^2}(u_{xx})$

The equation now becomes

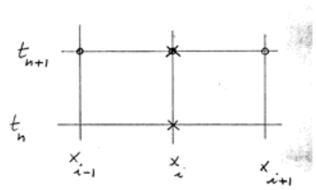
$$\frac{u_{i,n+1} - u_{i,n}}{\Delta t} = \frac{u_{i-1,n+1} - 2u_{i,n+1} + u_{i+1,n+1}}{(\Delta x)^2}$$



Defining $\lambda = \Delta t / \Delta x^2$, the above equation can be rewritten as

$$-\lambda u_{i-1,n+1} + (1+2\lambda)u_{i,n+1} - \lambda u_{i+1,n+1} = u_{i,n}$$

IC and BC are the same as before.



$$u_{0,n+1} = g_0(n+1)$$

$$u_{M,n+1} = g_1(n+1)$$

$$u_{i,0} = f(x_i)$$

Equations similar to the above should be written for each grid point

$$1 \le i \le M-1$$

Note: left boundary has i=0 and right boundary has i=M, thus total of (M+1) grid points (also knows as nodes) are present.
Thus we have (M-1) linear simultaneous equations with (M-1) unknowns.
Explicit solution is not possible.

Convergence of Implicit form

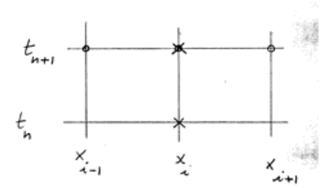
Can show using Taylor series

$$\frac{u_{i,n+1} - u_{i,n}}{\Delta t} = \frac{u_{i-1,n+1} - 2u_{i,n+1} + u_{i+1,n+1}}{(\Delta x)^2} + z_{i,n}$$

Where

$$z_{i,n} = \underbrace{\left[\frac{\Delta t}{2} u_{tt} + \frac{(\Delta x)^2}{12} u_{xxxx}\right]}_{\text{leading terms in truncation error}} + O\left[(\Delta t)^2\right] + O\left[(\Delta x)^4\right]$$

From the leading terms in the above



$$z_{i,n} = O\left[\Delta t + (\Delta x)^2 \right]$$

It can be shown that implicit method converges to the exact solution of the PDE as $\Delta t \to 0$ and $\Delta x \to 0$ for any value of $\dfrac{\Delta t}{(\Delta x)^2}$

The difference equation is now written for $1 \le i \le M-1$ as follows.

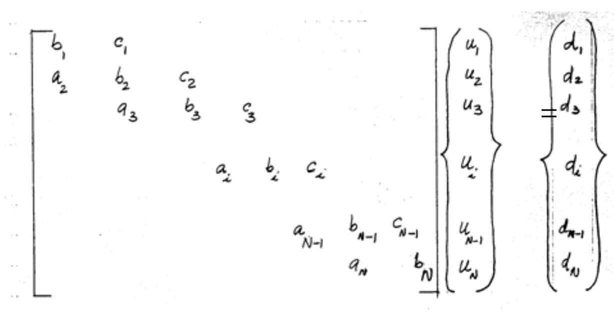
$$(1+2\lambda)u_{1,n+1} - \lambda u_{2,n+1} = u_{1,n} + \lambda g_0(t_{n+1}) \qquad \text{for } i = 1$$

$$-\lambda u_{i-1,n+1} + (1+2\lambda)u_{i,n+1} - \lambda u_{i+1,n+1} = u_{i,n} \qquad \text{for } 2 \le i \le M-2$$

Finally, (show M, N in figure)

$$-\lambda u_{M-2,n+1} + (1+2\lambda)u_{M-1,n+1} = u_{M-1,n} + \lambda g_1(t_{n+1})$$

for i = M



Note: In the matrix shown above, change in subscript N=M-1 is adopted for simplicity.

The RHS terms $d_1, d_2, \ldots d_N$ are known quantities. All matrix elements not shown are zero.
The matrix above is called a tridiagonal matrix i.e., only sub-diagonal, diagonal, and super diagonal terms are non–zero.

Solution Procedure

Solution can be obtained by Gauss elimination procedure

Recursion Relation

$$u_i = \gamma_i - \frac{c_i}{\beta_i} u_{i+1}$$

Constants β_i and γ_i are to be determined.

Substituting into the i[th] equation of the set for u_{i-1} gives

$$a_i\left(\gamma_{i-1}-\frac{c_{i-1}}{\beta_{i-1}}u_i\right)+b_iu_i+c_iu_{i+1}=d_i$$

Rewrite as

$$u_i=\frac{d_i-a_i\gamma_{i-1}}{b_i-\dfrac{a_ic_{i-1}}{\beta_{i-1}}}-\frac{c_iu_{i+1}}{b_i-\dfrac{a_ic_{i-1}}{\beta_{i-1}}}$$

$$u_i=\gamma_i-\frac{c_i}{\beta_i}u_{i+1}$$

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Comparing the two equations we have recursion relations for β and γ

$$\beta_i=b_i-\frac{a_ic_{i-1}}{\beta_{i-1}}$$

$$u_i=\gamma_i-\frac{c_i}{\beta_i}u_{i+1}$$

$$\gamma_i=\frac{d_i-a_i\gamma_{i-1}}{\beta_i}$$

From the first equation (see matrix in slide 10)

$$u_1=\frac{d_1}{b_1}-\frac{c_1}{b_1}u_2$$

Where $\beta_1=b_1$ and $\gamma_1=\dfrac{d_1}{\beta_1}$

$$u_i = \gamma_i - \frac{c_i}{\beta_i} u_{i+1}$$

From the last equation (see matrix in slide 10)

$$u_N = \frac{d_N - a_N u_{N-1}}{b_N} = \frac{d_N - a_N \left(\gamma_{N-1} - \frac{c_{N-1}}{\beta_{N-1}} u_N \right)}{b_N}$$

Rearranging yields

$$u_i = \gamma_i - \frac{c_i}{\beta_i} u_{i+1}$$

$$u_N = \frac{d_N - a_N \gamma_{N-1}}{b_N - \frac{a_N c_{N-1}}{\beta_{N-1}}} = \gamma_N$$

Algorithm summary:

Recursion formulas for β_i and γ_i

$$\beta_1 = b_1, \ \gamma_1 = \frac{d_1}{\beta_1}$$

$$\beta_i = b_i - \frac{a_i c_{i-1}}{\beta_{i-1}}, \qquad i = 2, 3, \ldots, N$$

$$\gamma_i = \frac{d_i - a_i \gamma_{i-1}}{\beta_i}, \qquad i = 1, 2, 3, \ldots, N$$

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Once the coefficients have been calculated, the solution vector u can be calculated starting with u_N and going backwards, as follows:

$$u_N = \gamma_N$$

$$u_i = \gamma_i - \frac{c_i u_{i+1}}{\beta_i}, \qquad i=N-1, N-2, \ldots, 1$$

Note: n denotes time level and N denotes the last but one node in the i-direction.

Recursion formulas for β_i and γ_i

$$\beta_1 = b_1,\ \gamma_1 = \frac{d_1}{\beta_1}$$

$$\beta_i = b_i - \frac{a_i c_{i-1}}{\beta_{i-1}}, \qquad i = 2, 3, \ldots, N$$

$$\gamma_i = \frac{d_i - a_i \gamma_{i-1}}{\beta_i}, \qquad i = 1, 2, 3, \ldots, N$$

Gauss elimination can cause large round-off errors.
Implicit schemes usually require more computational steps, but the ratio $\Delta t/(\Delta x)^2$ has no restrictions, is a definite advantage.

Stability (7.10)

A finite difference form is convergent if the solution tends to the exact solution as $(\Delta t, \Delta x) \to 0$ (in the absence of round off error).

Stability refers to amplification of information present in IC, BC or introduced by errors in the numerical procedure such as round off error.

Von-Neumann's stability analysis:
Stability implies only boundedness, not the magnitude of deviation from the true solution.

Key features of stability analysis:
Assume that: i) At any stage, t=0 here, a Fourier expansion can be made of some initial function f(x), and a typical term in the expansion can be written as $e^{j\beta x}$ where β is a positive constant and $j = \sqrt{-1}$

ii) Separation of time and space dependence can be made.

At time t, the term becomes $\psi(t)e^{j\beta x}$

By substituting in the difference equation, the form of ψ(t) can be determined and stability criterion established.

Example

Explicit finite difference form

$$\frac{u_{i,n+1} - u_{i,n}}{\Delta t} = \frac{u_{i-1,n} - 2u_{i,n} + u_{i+1,n}}{(\Delta x)^2}$$

Substitute for each u.

$$\frac{\psi(t+\Delta t)e^{j\beta x} - \psi(t)e^{j\beta x}}{\Delta t} = \frac{\psi(t)}{(\Delta x)^2}\left[e^{j\beta(x-\Delta x)} - 2e^{j\beta x} + e^{j\beta(x+\Delta x)}\right]$$

Cancel exp (jβx) throughout

$$\psi(t + \Delta t) = \psi(t)\left[1 + \lambda(e^{-j\beta\Delta x} - 2 + e^{j\beta\Delta x})\right]$$

Trig. identity: $Cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$ can be used to get

$$\psi(t + \Delta t) = \psi(t)\left[1 + \lambda(-2 + 2\cos(\beta\Delta x))\right]$$

Since $\cos\theta = 1 - 2\sin^2(\theta/2)$

$$\psi(t+\Delta t) = \psi(t)\left[1 - 2\lambda\left(2\sin^2\frac{\beta\Delta x}{2}\right)\right]$$

$$\psi(t+\Delta t) = \psi(t)\left[1 - 4\lambda\sin^2\left(\frac{\beta\Delta x}{2}\right)\right]$$

If we choose $\psi(0) = 1$, this has the solution

$$\psi(t) = \left(1 - 4\lambda\sin^2(\beta\Delta x/2)\right)^{(t/\Delta t)}$$

And can be proven by substitution (see next slide).

$$\psi(t+\Delta t) = \left(1 - 4\lambda\sin^2(\beta\Delta x/2)\right)^{(t+\Delta t)/\Delta t}$$

$$= \psi(t)\left[1 - 4\lambda\sin^2(\beta\Delta x/2)\right]$$

For stability, ψ(t) must be bounded as $(\Delta t, \Delta x) \to 0$

This requires

$$\left|1 - 4\lambda\sin^2(\beta\Delta x/2)\right| \le 1$$

An amplification factor ξ is usually defined as follows:

$$\xi = 1 - 4\lambda \sin^2(\beta \Delta x / 2)$$

Which shows $|\xi| \le 1$ for stability.

In the Fourier expansion we considered only one term corresponding to 1 value of β. When all possible values of β are considered sin(βΔx/2) could become 1.

Therefore, the stability condition becomes

$$\lambda = \frac{\Delta t}{(\Delta x)^2} \le \frac{1}{2}$$

λ = 0.5, |ξ| = 1
λ = 1.0, |ξ| = 3 (unstable)

Intuitively it implies that $u_{i,n}$ affects $u_{i,n+1}$ in a "non-negative" manner.

A similar analysis for the implicit method would give

$$\xi = \frac{1}{1+4\lambda \sin^2(\beta \Delta x / 2)}$$

Since $\xi \le 1$ for all λ, the procedure is unconditionally stable.

<u>Consistency</u> means that the procedure may in fact approximate the solution of the PDE under study and not the solution of some other PDE.