AE/ME 339

Computational Fluid Dynamics (CFD)

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Pressure correction method (6.8)

Relaxation method is well known for solving elliptic equations.
It is an iterative procedure.
Many flow problems are elliptic-parabolic in nature.
Pressure correction techniques have been developed for such flows.
Patankar and Spalding developed the SIMPLE (Semi-Implicit Method for Pressure-Linked Equations) algorithm which uses pressure correction method.

Continuity Equation for incompressible flow: $\bar{\nabla} \cdot \bar{V} = 0$

$$\bar{\nabla} \cdot \bar{V} = \frac{\partial u}{\partial x} + \frac{\partial \text{v}}{\partial y} + \frac{\partial w}{\partial z} = 0 \text{....................(6.72)}$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial}{\partial y}\left(\frac{\partial \text{v}}{\partial x} + \frac{\partial u}{\partial y}\right) + \mu \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \rho f_x \text{..............(6.69)}$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \mu \frac{\partial}{\partial x}\left(\frac{\partial \text{v}}{\partial x} + \frac{\partial u}{\partial y}\right) + 2\mu \frac{\partial^2 \text{v}}{\partial y^2} + \mu \frac{\partial}{\partial z}\left(\frac{\partial \text{v}}{\partial z} + \frac{\partial w}{\partial y}\right) + \rho f_y \text{..............(6.70)}$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \mu \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \mu \frac{\partial}{\partial y}\left(\frac{\partial \text{v}}{\partial z} + \frac{\partial w}{\partial y}\right) + 2\mu \frac{\partial^2 w}{\partial z^2} + \rho f_z \text{..............(6.71)}$$

$$\frac{\partial u}{\partial x} = -\frac{\partial \text{v}}{\partial y} - \frac{\partial w}{\partial z}$$

Differentiate w.r.t. x

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 \text{v}}{\partial y \partial x} - \frac{\partial^2 w}{\partial z \partial x}$$

Add $\frac{\partial^2 u}{\partial x^2}$ on both sides of the above equation and
and multiply by μ throughout to get the following

$$2\mu \frac{\partial^2 u}{\partial x^2} = \mu \frac{\partial^2 u}{\partial x^2} - \mu \frac{\partial^2 \text{v}}{\partial y \partial x} - \mu \frac{\partial^2 w}{\partial z \partial x}$$

Substitute in Eq. 6.69 for the 2nd term on the RHS using the above to get the following after expanding the terms and canceling terms (next slide)

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After performing the manipulations described in Section 6.8.1, we get the equations in the following form, where note that:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\overline{\nabla} \cdot \overline{V} = 0 \text{............................}(6.77)$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u + \rho f_x \text{..................}(6.78)$$

$$\rho \frac{D\text{v}}{Dt} = -\frac{\partial p}{\partial y} + \mu \nabla^2 \text{v} + \rho f_y \text{..................}(6.79)$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \mu \nabla^2 w + \rho f_z \text{..................}(6.80)$$

In the above equations, the dependent variables are u, v, w, and p.
Note that the energy equation is not present in the set.
This is a consequence of assuming ρ = constant and μ = constant.
The energy equation is decoupled from the continuity and momentum equations, and it needs to be solved only for non-isothermal cases.

Solution techniques developed for compressible flows are not suitable for incompressible flows. Why?
Let us examine the stability criterion of Tannehill, Anderson and Pletcher (*Computational Fluid Flow and Heat Transfer*, 2nd Edition, McGraw-Hill).

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$$\Delta t \le \frac{1}{\frac{|u|}{\Delta x} + \frac{|v|}{\Delta y} + a\sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}}} \quad \text{.............(6.81)}$$

Note that the speed of sound expression is as follows:

$$a^2 = (\partial p / \partial \rho)_s$$

Which shows that $a = \infty$ for incompressible substances.

Therefore, if the time-dependent technique of compressible flow is used, Δt should be equal to zero.
Thus the need for other techniques for incompressible flows.
The SIMPE algorithm has been initially developed for incompressible flows and later extended to compressible flows.
To illustrate the method, consider a 2D flow.

Continuity: $$\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} = 0$$

Using central difference (CD) this yields

$$\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{\mathrm{v}_{i,j+1} - \mathrm{v}_{i,j-1}}{2\Delta y} = 0 \qquad (6.83)$$

Substitution shows that the velocity pattern shown in Fig. 6.13 (see next page) satisfies Eq. 6.83.
Yet a real flow field would not behave in such a physically unrealistic manner.

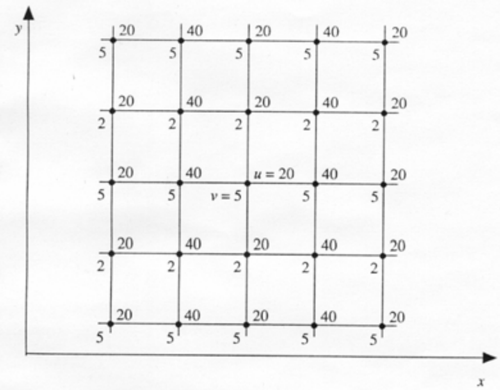

FIG. 6.13
Discrete checkerboard velocity distribution at each grid point; the number at the upper : that at the lower left is *v*.

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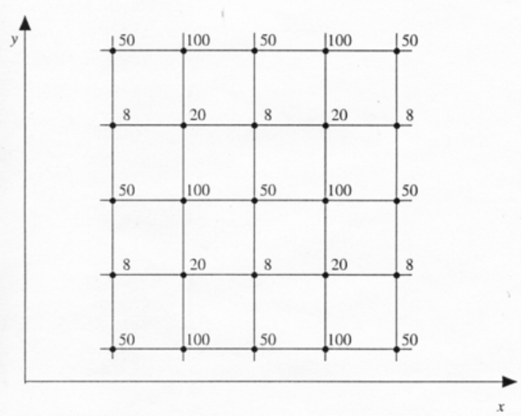


FIG. 6.14
Discrete checkerboard pressure distribution.

Note the zig-zag distributions of both the u and v velocity components in Figure 6.13 would satisfy the central difference form of the continuity equation.

Zig-zag pressure distribution is seen in Figure 6.14. Central difference form of first derivative would show zero pressure gradient for this distribution.

Now consider the pressure gradient terms in the momentum equation

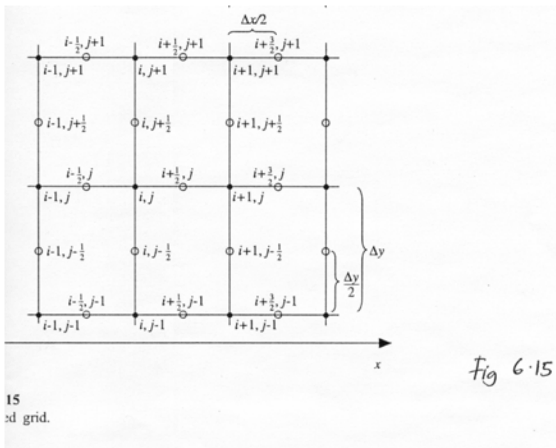
$$\frac{\partial p}{\partial x} = \frac{p_{i+1,j} - p_{i-1,j}}{2\Delta x}$$

$$\frac{\partial p}{\partial y} = \frac{p_{i,j+1} - p_{i,j-1}}{2\Delta y}$$

These equations yield $\frac{\partial p}{\partial x} = 0, \frac{\partial p}{\partial y} = 0$

for the zig-zag pressure distribution of Fig. 6.14.
Numerical solution will yield a uniform pressure field, washing out the actual distribution.
Conclusion: Mere CD formulation may not be suitable for incompressible flow.

The staggered grid shown in Figure 6.15 has been proposed to eliminate the difficulties of central differencing.



In this method, the pressure are calculated at the grid points (i-1, j), (i, j), (i+1, j), (i, j+1), (i, j-1), etc., shown as solid circles.
The velocities are calculated at (i-1/2, j), (i+1/2, j), (i, j+1/2), (i, j-1/2), etc.
Because of using the staggered grid, oscillations often wash out.

Much of the benefits of using staggered grid have been discovered in applications, even though theoretical explanations often fall short.

The continuity equation becomes

$$\frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x} + \frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta y} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(6.85)$$

Adjacent velocity points are used in the continuity equation, also leading to less oscillations.

Underlying Rationale (6.8.3)

Uses an iterative procedure and uses some interesting heuristic reasoning for the iteration procedure. The following steps are involved.

1. Guess a pressure field, p*.
2. Solve momentum equations for u*, v*, w* using p*.
3. Use the velocity field from Step 2 to calculate a pressure correction p'. The corrected pressure p = p*+p'.
4. Calculate velocity corrections u', v' and w' using p.'
   i.e., u = u*+u', etc.
5. Replace p* in Step 1 with p, and repeat Steps 2-4 till the velocity field does satisfy the continuity equation.