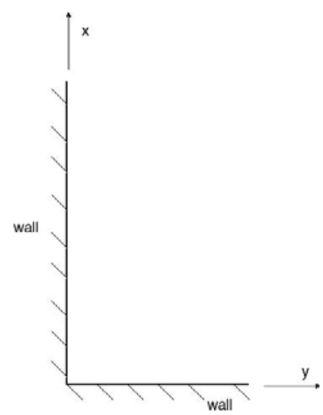
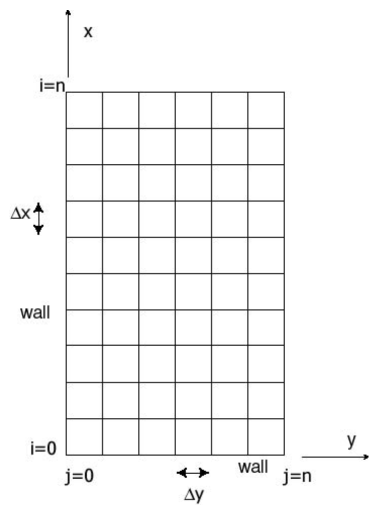
AE/ME 339 Computational Fluid Dynamics (CFD)

K. M. Isaac

Let us consider flow over a heated vertical wall as shown in figure.
A free convection flow induced by buoyancy forces is established close to the wall.

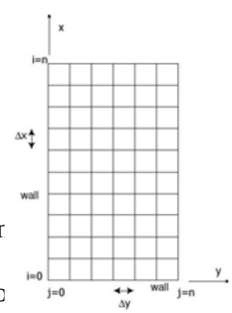




Boundary conditions:

At $t = 0$:
$u = v = 0$, $T = T_{inf}$ (everywhere in the solution domain)

At $t > 0$:
$y = 0$: $u = v = 0$, $T = T_w$ (heated wall conditior
$y = inf$: $u = 0$, $T = T_{inf}$ (far field condition)
$x = 0$: $u = v = 0$, $T = T_{inf}$ (bottom wall conditic



Governing equations

$$\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}=0.............(1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+g_x\beta(T-T_\infty)+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)..............(2)$$

$$\frac{\partial \mathrm{v}}{\partial t}+u\frac{\partial \mathrm{v}}{\partial x}+\mathrm{v}\frac{\partial \mathrm{v}}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+g_y\beta(T-T_\infty)+\nu\left(\frac{\partial^2 \mathrm{v}}{\partial x^2}+\frac{\partial^2 \mathrm{v}}{\partial y^2}\right)..............(3)$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+\mathrm{v}\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{\mu}{\rho c_p}\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial \mathrm{v}}{\partial y}\right)^2+\left(\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}\right)^2\right]..............(4)$$

The Navier-Stokes equations derived in Chapter 2 can be simplified using the boundary layer assumptions.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.............(1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g_x\beta(T-T_\infty)+\nu\frac{\partial^2 u}{\partial y^2}..............(5)$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}...............(6)$$

Note the neglected terms in the above equations

Eq. 2: 2nd derivative of u w.r.t x is neglected because it is small compared to the other term.
Pressure gradient is assumed to be zero.

Eq. 3: This equation is not solved in the present formulation
All the terms on the RHS are assumed to be zero.
Since p is assumed to be a constant, there is no need to use this equation.

Eq. 4: 2nd derivative of temperature w.r.t x and the viscous dissipation terms are neglected.

The simplified equations (continuity, x-momentum and energy) can be non-dimensionalized as follows:

$$\xi = x(g\beta\Delta T/\nu^2)^{\frac{1}{3}}$$

$$\eta = y(g\beta\Delta T/\nu^2)^{\frac{1}{3}}$$

$$\tau = t(g\beta\Delta T)^{\frac{2}{3}}/\nu^{\frac{1}{3}}$$

$$\overline{u} = u/(\nu g\beta\Delta T)^{\frac{1}{3}}$$

$$\overline{\mathrm{v}} = \mathrm{v}/(\nu g\beta\Delta T)^{\frac{1}{3}}$$

$$\theta = \frac{T - T_\infty}{T_1 - T_\infty} = \frac{T - T_\infty}{\Delta T}$$

Note that g_x is the magnitude of the gravity component in the x-direction. β is the volumetric coefficient of thermal expansion defined as

$$\beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}$$

Further, we assume that the Prandtl number Pr (= μcp/k= ν/ α) = 1. The equations will now be obtained in a form that is easy to solve numerically.

Note β has units of (1/K)

If the assumption Pr = 1 is not made, the RHS of the energy equation will have a Pr factor.

The energy equation for non-unity Prandtl number can be differenced using the explicit formulation as follows

$$\frac{\theta_{i,j}^{n+1} - \theta_{i,j}^{n}}{\Delta \tau} + \overline{u}_{i,j}^{n} \frac{\theta_{i,j}^{n} - \theta_{i-1,j}^{n}}{\Delta \xi} + \overline{\text{v}}_{i,j}^{n} \frac{\theta_{i,j+1}^{n} - \theta_{i,j}^{n}}{\Delta \eta} = \frac{1}{\Pr}\left(\frac{\theta_{i,j+1}^{n} - 2\theta_{i,j}^{n} + \theta_{i,j-1}^{n}}{(\Delta \eta)^2}\right) \ldots\ldots\ldots(7)$$

Note the use of BD for the second term on the LHS and FD for the third term and CD for the RHS term.

The x-momentum equation can now be written in the finite difference form as

$$\frac{\overline{u}_{i,j}^{n+1}-\overline{u}_{i,j}^{n}}{\Delta\tau}+\overline{u}_{i,j}^{n}\frac{\overline{u}_{i,j}^{n}-\overline{u}_{i-1,j}^{n}}{\Delta\xi}+\overline{\mathrm{v}}_{i,j}^{n}\frac{\overline{u}_{i,j+1}^{n}-\overline{u}_{i,j}^{n}}{\Delta\eta}$$

$$=\theta_{i,j}^{n+1}+\frac{\overline{u}_{i,j+1}^{n}-2\overline{u}_{i,j}^{n}+\overline{u}_{i,j-1}^{n}}{(\Delta\eta)^{2}}\text{...........................}(8)$$

The continuity equation becomes

$$\frac{\overline{u}_{i,j}^{n+1}-\overline{u}_{i-1,j}^{n+1}}{\Delta\xi}+\frac{\overline{\mathrm{v}}_{i,j}^{n+1}-\overline{\mathrm{v}}_{i,j-1}^{n+1}}{\Delta\eta}=0\text{...........}(9)$$

Equations 7, 8 and 9 are solved sequentially for each time step. Note that θ^{n+1} is known once the energy equation is solved.

Therefore, in the momentum equation, it can be treated as a known quantity.

Since u is known from solving the x-momentum equation, v^{n+1} are the only unknowns in Eq. (9).

Thus these set of equations can be solved explicitly.