AE/ME 339 Computational Fluid Dynamics (CFD)

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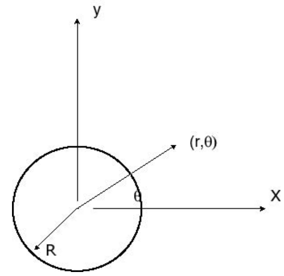
Flow over a cylinder is a classical problem
Potential flow over a cylinder is equivalent to a doublet in uniform flow
See potential flow theory for details.
The analytical solution for ϕ is as follows



$$\phi = \underbrace{U_\infty r\cos(\theta)}_{\text{uniform flow}} + \underbrace{U_\infty \frac{R^2}{r}\cos(\theta)}_{\text{doublet flow}} = U_\infty r\cos(\theta)\left(1+\frac{R^2}{r^2}\right)$$

Velocity component at any point is given by

$$\overline{V} = \hat{e}_r u_r + \hat{e}_\theta u_\theta = \nabla\phi \equiv \hat{e}_r \frac{\partial\phi}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

ur =

Differentiation yields

$$u_r = U_\infty \cos(\theta)\left(1 - \frac{R^2}{r^2}\right)$$

$$u_\theta = -U_\infty \sin(\theta)\left(1 + \frac{R^2}{r^2}\right)$$

Pressure distribution can be represented in terms of the pressure Coefficient C_p

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2}$$

Bernoulli's equation

$$p + (1/2)\rho V^2 = p_\infty + (1/2)\rho U_\infty^2$$

$$\therefore C_p = 1 - \left(\frac{V}{U_\infty}\right)^2$$

Recall

$$V^2 = u_r^2 + u_\theta^2$$

Surface pressure distribution can now be obtained by substituting r=R

$$u_r\big|_{r=R} = 0$$

$$u_\theta\big|_{r=R} = -2U_\infty \sin(\theta)$$

$$C_p\big|_{r=R} = 1 - 4\sin^2(\theta)$$

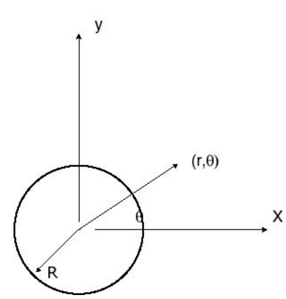
Note that the surface pressure distribution is independent of the cylinder radius.

Flow Over a Cylinder

The transformation from (x,y) to polar coordinates (r, θ) will give boundary-fitted coordinates. (Recall r = R (constant) is the cylinder surface.

Here the domain is between r = R and r = infinity.

Computationally, the outer boundary needs to be finite, say r = r_1, where the far field (free-stream) conditions can be applied.



We can also use the transformation σ = 1/r. With this tranformation, σ = 1/R represents the cylinder surface and σ = 0 represents conditions at infinity.
In the new (σ, θ) system, the computational space lies between σ = 0 (far field) and 1/R (surface).

First consider the polar coordinates. The relationship between the two systems is as follows:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Transformation metrics:

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These transformations can be checked by applying them to Laplace's equation, which in polar coordinates is given by

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}=0.......(1)$$

$$\frac{\partial x}{\partial r}=\cos(\theta),\frac{\partial^2 x}{\partial r^2}=0$$
$$\frac{\partial x}{\partial\theta}=-r\sin(\theta),\frac{\partial^2 x}{\partial\theta^2}=-r\cos(\theta)$$
$$\frac{\partial y}{\partial r}=\sin(\theta),\frac{\partial^2 y}{\partial r^2}=0$$
$$\frac{\partial y}{\partial\theta}=r\cos(\theta),\frac{\partial^2 y}{\partial\theta^2}=-r\sin(\theta)$$

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General form of the transformation

$$\frac{\partial^2}{\partial r^2} = \left(\frac{\partial^2 x}{\partial r^2}\right)\frac{\partial}{\partial x} + \left(\frac{\partial^2 y}{\partial r^2}\right)\frac{\partial}{\partial y} + \left(\frac{\partial x}{\partial r}\right)^2 \frac{\partial^2}{\partial x^2} +$$

$$\left(\frac{\partial y}{\partial r}\right)^2 \frac{\partial^2}{\partial y^2} + 2\left(\frac{\partial x}{\partial r}\right)\left(\frac{\partial y}{\partial r}\right)\frac{\partial^2}{\partial x \partial y}$$

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Substituting for the metrics give the following

$$\frac{\partial^2}{\partial r^2} = \underbrace{\left(\frac{\partial^2 x}{\partial r^2}\right)}_{0}\frac{\partial}{\partial x} + \underbrace{\left(\frac{\partial^2 y}{\partial r^2}\right)}_{0}\frac{\partial}{\partial y} + \underbrace{\left(\frac{\partial x}{\partial r}\right)^2}_{\cos^2(\theta)} \frac{\partial^2}{\partial x^2} +$$

$$\underbrace{\left(\frac{\partial y}{\partial r}\right)^2}_{\sin^2(\theta)} \frac{\partial^2}{\partial y^2} + 2\underbrace{\left(\frac{\partial x}{\partial r}\right)}_{\cos(\theta)}\underbrace{\left(\frac{\partial y}{\partial r}\right)}_{\sin(\theta)}\frac{\partial^2}{\partial x \partial y}$$

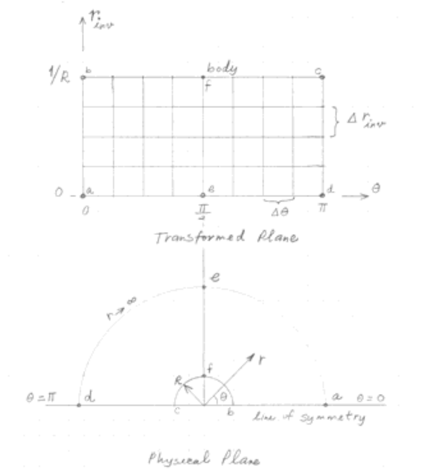
Similarly, get expressions for the other terms

$$\frac{\partial^2}{\partial\theta^2}, \frac{\partial}{\partial r}$$

And substitute in Laplace's equation in polar coordinates to transoform The equations to cartesian coordinates.

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0.......(2)$$

For this problem the transformation $\sigma = 1/r$ can be used to transform the original equation in terms of σ.
Also since the flow is symmetric about $\theta = 0$ line, we need to solve only one half of the flow domain.



The governing equation can now be written in terms of σ as follows

$$\frac{\partial}{\partial r} = \frac{\partial \sigma}{\partial r}\frac{\partial}{\partial \sigma} = -\sigma^2 \frac{\partial}{\partial \sigma}$$

$$\frac{\partial^2}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\sigma^2 \frac{\partial}{\partial \sigma}\right) =$$

$$-\sigma^2 \frac{\partial}{\partial \sigma}\left(-\sigma^2 \frac{\partial}{\partial \sigma}\right) = \sigma^4 \frac{\partial^2}{\partial \sigma^2} + 2\sigma^3 \frac{\partial}{\partial \sigma}$$

Using the above relations in Laplaces equation gives

$$\sigma^2 \frac{\partial^2 \varphi}{\partial \sigma^2} + \sigma \frac{\partial \varphi}{\partial \sigma} + \frac{\partial^2 \varphi}{\partial \theta^2} = 0$$

The above equation needs to be solved on the rectangular grid shown
In the figure.
Before starting the solution, the boundary conditions also must be
Transformed.

At r = R: $r = R:$

$$\frac{\partial \phi}{\partial r} = u_r = 0$$

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=R} = -\sigma^2 \left.\frac{\partial \phi}{\partial \sigma}\right|_{\sigma=(1/R)} = 0$$

$$\therefore \left.\frac{\partial \phi}{\partial \sigma}\right|_{\sigma=(1/R)} = 0$$

$$r \to \infty :$$

$$\phi = U_\infty r \cos(\theta)$$

$$r \to \infty : \frac{\partial \phi}{\partial r} = -\sigma^2 \frac{\partial \phi}{\partial \sigma} = U_\infty \cos(\theta)$$

$$or : \phi = \frac{U_\infty}{\sigma} \cos(\theta)$$

Along the line of symmetry $u_\theta = 0$.

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

$$i.e., \frac{\partial \phi}{\partial \theta} = 0$$