AE/ME 339 Computational Fluid Dynamics (CFD)

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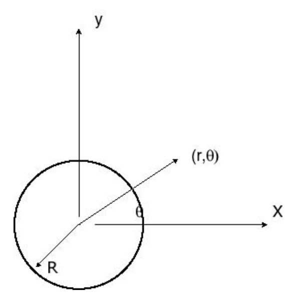
Relaxation technique is suitable for solving elliptic equations such as the steady state heat conduction equation and incompressible potential flow equation.

Note that we solved the heat conduction equation as an unsteady problem in which case the equation was parabolic.
Steady state solution in that case was obtained as an asymptote.



Relaxation technique is an iterative technique particularly suited for elliptic partial differential equations.
Examples are:

- Steady state heat conduction problem
- Incompressible steady flow problems

It can be explicit or implicit
Here we will use an explicit technique known as *point-iterative* method.
Consider a 2D, steady, incompressible, *irrotational* flow.
The governing equation can be written in cartesian coordinates in terms of the velocity potential ϕ.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 ....(6.31)$$

Central differencing gives the following finite difference form

$$\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{(\Delta x)^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{(\Delta y)^2} = 0 ......(6.32)$$

If we are solving the equation in a domain shown in Figure 6.4, the boundary conditions must be specified on the four sides of the domain, since the equation is elliptic.
The above equation can be written for each of the interior grid points which will give a set of coupled equations whose solution can be obtained by a suitable method.
Relaxation technique avoids using solution of simultaneous equations.

$$\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{(\Delta x)^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{(\Delta y)^2} = 0......(6.32)$$

If we treat only $\phi_{i,j}$ as the unknown, the rest of the terms assumed to be known from the previous iterations, then an explicit solution is possible.
If we represent the iteration level by n, then the following equation can be written using Eq. (6.32)

$$\phi_{i,j}^{n+1} = \frac{(\Delta x)^2(\Delta y)^2}{2\left[(\Delta x)^2 + (\Delta y)^2\right]}\left\{\frac{\phi_{i+1,j}^n + \phi_{i-1,j}^n}{(\Delta x)^2} + \frac{\phi_{i,j+1}^n + \phi_{i,j-1}^n}{(\Delta y)^2}\right\}......(6.33)$$

For starting the solution, initial guesses are made, such as a uniform flow at the interior points.
After solution is completed at all the interior points, the next iteration step begins.
The values at a node is updated as soon as it becomes available.
This procedure speeds up convergence.
Iteration is terminated when a specified convergence criterion is satisfied.

Successtive Over-Relaxation (SOR)

A modification of the method is called successive over-relaxation, explained below. Define the following intermediate value.

$$\bar{\phi}_{i,j}^{n+1} = \frac{(\Delta x)^2(\Delta y)^2}{2\left[(\Delta x)^2 + (\Delta y)^2\right]}\left\{\frac{\phi_{i+1,j}^n + \phi_{i-1,j}^{n+1}}{(\Delta x)^2} + \frac{\phi_{i,j+1}^n + \phi_{i,j-1}^{n+1}}{(\Delta y)^2}\right\}......(6.36)$$

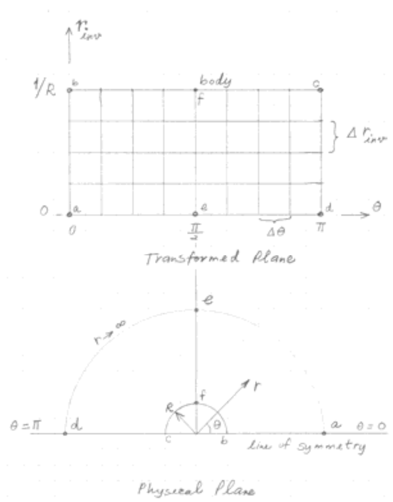
Note that since we are sweeping from left to right and bottom to top, the $\phi_{i-1,j}^{n+1}$ and $\phi_{i,j-1}^{n+1}$ values are known.
Now extrapolate to get the (n+1) value as follows.

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n} + \omega(\overline{\phi}_{i,j}^{n+1} - \phi_{i,j}^{n}) \ldots\ldots(6.37)$$

Note that $\omega = 1$ yields the original method. For SOR, ω usually lies in the following range.

$$1 \le \omega \le 2$$

Some times under-relaxation ($\omega<1$) is used when oscillations are observed.



The equation can first be non-dimensionalized as follows:

$$\bar{\phi} = \frac{\phi}{RU_\infty}, \bar{r} = \frac{r}{R}$$

$$\bar{u}_r = \frac{u_r}{U\infty}, \bar{u}_\theta = \frac{u_\theta}{U\infty}$$

The transformed equation becomes

$$\frac{\partial^2 \bar{\phi}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{\phi}}{\partial \bar{r}} + \frac{1}{\bar{r}^2} \frac{\partial^2 \bar{\phi}}{\partial \theta^2} = 0$$

Boundary conditions:
At the surface

$$\frac{\partial \bar{\phi}}{\partial \bar{r}} = 0$$

At infinity

$$\phi = U_\infty r \cos(\theta)$$

$$\left(\frac{\phi}{RU_\infty}\right) RU_\infty = U_\infty \frac{r}{R} R \cos(\theta)$$

$$\bar{\phi} = \bar{r} \cos(\theta)$$

Note that the problem is completely non-dimensionalized. R and U_∞ are absent in the equations and the boundary conditions.

For convenience, define a new variable σ = 1/r

$$\left[\sigma_{i,j}\right]^2\left[\frac{\phi_{i+1,j}-2\phi_{i,j}+\phi_{i-1,j}}{(\Delta\sigma)^2}\right]+\sigma_{i,j}\left[\frac{\phi_{i+1,j}-\phi_{i-1,j}}{2\Delta\sigma}\right]+\left[\frac{\phi_{i,j+1}-2\phi_{i,j}+\phi_{i,j-1}}{(\Delta\theta)^2}\right]=0$$

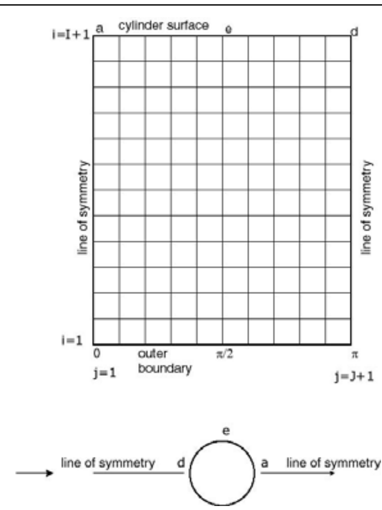
Nondimensionalizing is equivalent to choosing R = 1, and U_∞ = 1.

choosing R = 1, and U_∞ = 1

$$\phi_{i,j}\left[\frac{2(\sigma_{i,j})^2}{(\Delta\sigma)^2}+\frac{2}{(\Delta\theta)^2}\right]=\frac{(\sigma_{i,j})^2}{(\Delta\sigma)^2}\left[\phi_{i+1,j}+\phi_{i-1,j}\right]+$$

$$\frac{\sigma_{i,j}}{2\Delta\sigma}\left[\phi_{i+1,j}-\phi_{i-1,j}\right]+\frac{1}{(\Delta\theta)^2}\left[\phi_{i,j+1}+\phi_{i,j-1}\right]$$

$$\phi_{i,j} = \frac{1}{\left[\frac{2(\sigma_{i,j})^2}{(\Delta\sigma)^2} + \frac{2}{(\Delta\theta)^2}\right]} \left\{ \begin{array}{l} \frac{(\sigma_{i,j})^2}{(\Delta\sigma)^2}\left[\phi_{i+1,j} + \phi_{i-1,j}\right] + \\ \frac{\sigma_{i,j}}{2\Delta\sigma}\left[\phi_{i+1,j} - \phi_{i-1,j}\right] + \frac{1}{(\Delta\theta)^2}\left[\phi_{i,j+1} + \phi_{i,j-1}\right] \end{array} \right\}$$



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Solution by Relaxation technique

We'll denote the iteration step by a superscript (recall that previously we used superscript for time step which should not be confused with the present use of the superscript).
The following symbol notation will be be used in writing the algorithm

At iteration steps n and (n+1) we will write the dependent variable at node (i,j) as:

$$\phi_{i,j}^{n} \text{ and } \phi_{i,j}^{n+1}$$

We can now implement the SOR technique discussed earlier.

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Intermediate value is first calculated. Note the terms with superscript (n+1) on the RHS. Since we are sweeping from left to right and bottom to top these values are known at any given iteration step.

$$\bar{\phi}_{i,j}^{n+1} = c\left\{\begin{array}{l} a^2\left[\phi_{i+1,j}^{n} + \phi_{i-1,j}^{n+1}\right] + \\ \frac{a}{2}\left[\phi_{i+1,j}^{n} - \phi_{i-1,j}^{n+1}\right] + b^2\left[\phi_{i,j+1}^{n} + \phi_{i,j-1}^{n+1}\right] \end{array}\right\}$$

where

$$a = \frac{\sigma_{i,j}}{\Delta\sigma}, \quad b = \frac{1}{\Delta\theta}, \quad c = \frac{1}{2(a^2 + b^2)}$$

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SOR technique yields the following value at iteration step (n+1)

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n} + \omega(\overline{\phi}_{i,j}^{n+1} - \phi_{i,j}^{n}) \text{..........} (6.97)$$

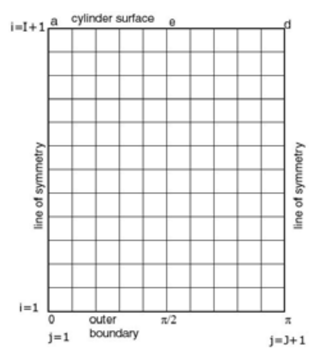
For SOR, ω should lie between 1 and 2.

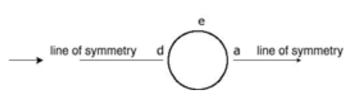
Boundary Conditions

There are three boundaries for the flow domain.

- cylinder surface
- far field conditions
- line of symmetry

For simplicity we will assume $U_\infty = 1$ and R = 1 (σ = 1).





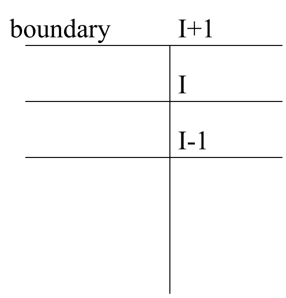
At the cylinder surface we have the flow tangency condition (recall that since we are solving potential flow, the tangential velocity is not zero. Only the normal velocity is zero). The condition can be written in terms of the velocity potential as

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=R} = 0$$

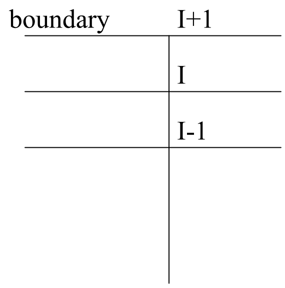
This transforms to

$$\left.\frac{\partial \phi}{\partial \sigma}\right|_{\sigma=(1/R)} = 0$$

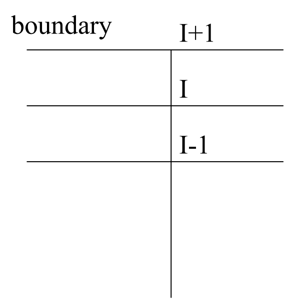
If we use a one-sided first order difference we get the following numerical equivalent of the the above BC



If we use 2nd order accurate derivative at the surface we can obtain the condition at the surface using the following Taylor series expansion



$$u_I = u_{I+1} - \left(\frac{\partial u}{\partial y}\right)_{I+1} \Delta y + \left(\frac{\partial^2 u}{\partial y^2}\right)_{I+1} \frac{(\Delta y)^2}{2!} -$$
$$\left(\frac{\partial^3 u}{\partial y^3}\right)_{I+1} \frac{(\Delta y)^3}{3!} + \ldots$$



Rearranging

$$\left(\frac{\partial u}{\partial y}\right)_{I+1} = \frac{u_{I+1} - u_I}{\Delta y} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{I+1} \frac{(\Delta y)}{2} -$$
$$\underbrace{\left(\frac{\partial^3 u}{\partial y^3}\right)_{I+1} \frac{(\Delta y)^2}{6} + \ldots}_{O[(\Delta y)^2]}$$

Using backward difference for the second derivative gives the following expression for the derivative

$$\left.\frac{\partial u}{\partial y}\right|_{I+1} = \frac{u_{I+1} - u_I}{\Delta y} + \frac{u_{I+1} - 2u_I + u_{I-1}}{(\Delta y)^2}\frac{\Delta y}{2} + O[(\Delta y)^2]$$

$$= \frac{3u_{I+1} - 4u_I + u_{I-1}}{2\Delta y}$$

For zero normal gradient at the boundary we get

$$u_{I+1} = \frac{4}{3}u_I - \frac{1}{3}u_{I-1}$$

or in terms of φ

$$\varphi_{I+1} = \frac{4}{3}\varphi_I - \frac{1}{3}\varphi_{I-1}$$

Similarly, for zero normal gradient at the left (i = 1) boundary we get

$$u_1 = \frac{4}{3}u_2 - \frac{1}{3}u_3$$

or in terms of φ

$$\varphi_1 = \frac{4}{3}\varphi_2 - \frac{1}{3}\varphi_3$$

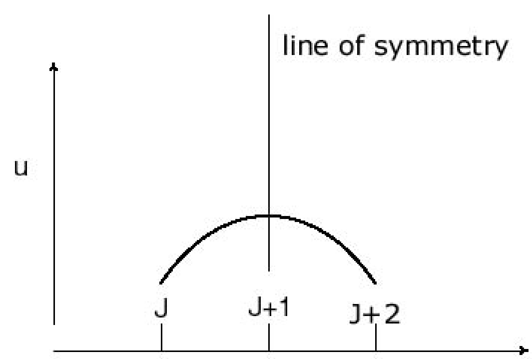
We obtained the following equation before

$$\bar{\phi}_{i,j}^{n+1} = c\left\{ \begin{array}{l} a^2\left[\phi_{i+1,j}^{n} + \phi_{i-1,j}^{n+1}\right] + \\ \frac{a}{2}\left[\phi_{i+1,j}^{n} - \phi_{i-1,j}^{n+1}\right] + b^2\left[\phi_{i,j+1}^{n} + \phi_{i,j-1}^{n+1}\right] \end{array} \right\}$$

where

$$a = \frac{\sigma_{i,j}}{\Delta\sigma}, \quad b = \frac{1}{\Delta\theta}, \quad c = \frac{1}{2(a^2 + b^2)}$$

Symmetry condition



Symmetry boundary conditon: $u_J = u_{J+2}$

Replace J+2 values with J values for right boundary

$$\bar{\phi}_{i,J+1}^{n+1} = c\left\{ \begin{array}{l} a^2\left[\phi_{i+1,J+1}^{n} + \phi_{i-1,J+1}^{n+1}\right] + \\ \frac{a}{2}\left[\phi_{i+1,J+1}^{n} - \phi_{i-1,J+1}^{n+1}\right] + b^2\left[\phi_{i,J}^{n} + \phi_{i,J}^{n+1}\right] \end{array} \right\}$$

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Replace 0 values with 2 values for the left boundary

$$\bar{\phi}_{i,1}^{n+1} = c\left\{ \begin{array}{l} a^2\left[\phi_{i+1,1}^{n} + \phi_{i-1,1}^{n+1}\right] + \\ \frac{a}{2}\left[\phi_{i+1,1}^{n} - \phi_{i-1,1}^{n+1}\right] + b^2(\phi_{i,2}^{n} + \phi_{i,0}^{n+1}) \end{array} \right\}$$

Since the $\phi_{i,0}^{n+1}$ is not known, it is replaced with $\phi_{i,2}^{n}$

$$\bar{\phi}_{i,1}^{n+1} = c\left\{ \begin{array}{l} a^2\left[\phi_{i+1,1}^{n} + \phi_{i-1,1}^{n+1}\right] + \\ \frac{a}{2}\left[\phi_{i+1,1}^{n} - \phi_{i-1,1}^{n+1}\right] + 2b^2\phi_{i,2}^{n} \end{array} \right\}$$

Convergence

The SOR method just described is also called Gauss-Seidal iteration method. Tannehill et al. (1997) have given the following convergence criterion for the method.
$a_{i,j}$ represents the coefficients.

$$\left|a_{i,i}\right| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|a_{i,j}\right| \qquad \text{for all } i$$

$$\left|a_{i,i}\right| > \sum_{\substack{j=1 \\ j \neq i}}^{n} \left|a_{i,j}\right| \qquad \text{for at least one } i$$

The above is a sufficient condition for convergence, meaning that even if the conditions are not met, convergence could be obtained.