Infinite-noise criticality: Nonequilibrium phase transitions in fluctuating environments

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How does temporal disorder (environmental noise) affect phase transitions in spreading and growth processes ?

- Stability criterion
- Logistic evolution equation
- Strong-noise renormalization group
- Contact process
- Monte-Carlo simulations

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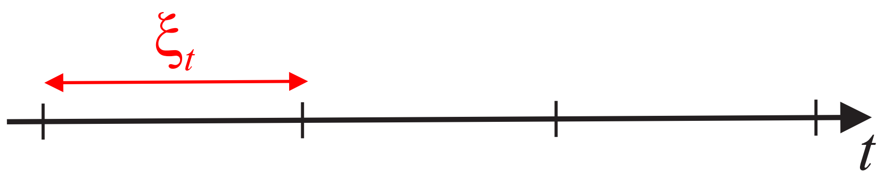
Stability of a critical point

spatial quenched disorder:

- system parameters **vary randomly in space** but are constant in time
- clean critical point is stable if $d\nu_\perp > 2$ (**Harris criterion**)

temporal disorder:

- system parameters **fluctuate randomly in time** but are uniform in space
- to derive stability criterion, divide time into intervals of length ξ_t
- fluctuations of distance from criticality between intervals:
$$\Delta r(\xi_t) \sim \xi_t^{-1/2}$$
- global distance from criticality:
$$r_{av} \sim \xi_t^{-1/\nu_\parallel}$$



Kinzel's stability criterion:

clean critical point stable if correlation time exponent fulfills $\nu_\parallel = \nu_\perp z > 2$

Generalization to arbitrary spatio-temporal disorder: T.V. + R. Dickman, PRE **93**, 032143 (2016)

- Stability criterion
- **Logistic evolution equation**
- Strong-noise renormalization group
- Contact process
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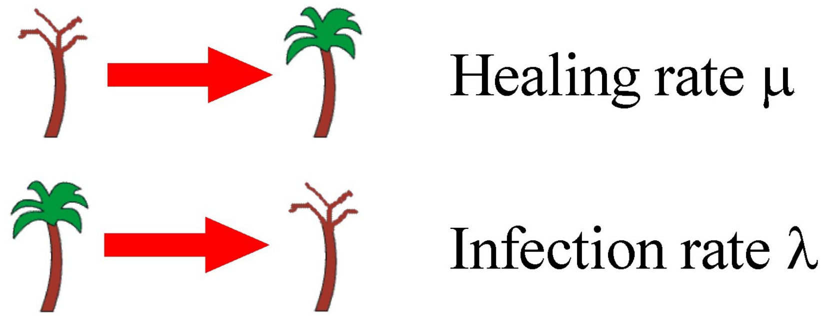
Logistic evolution equation

Single-variable (mean-field) model of epidemic spreading:

$$\frac{d}{dt}\rho(t) = (\lambda - \mu)\,\rho(t) - \lambda\rho^2(t)$$



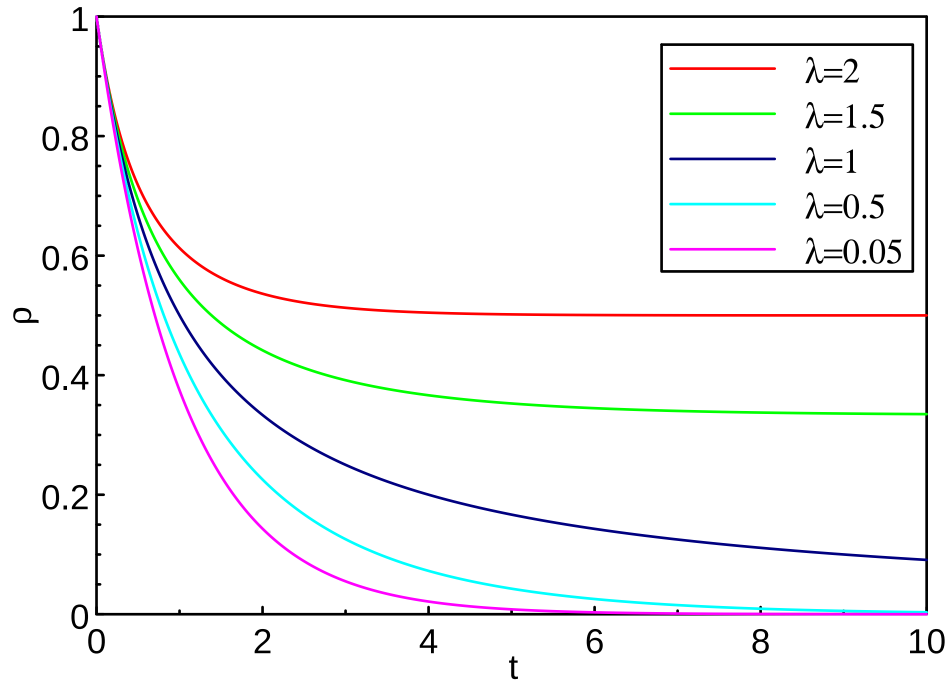
- ρ: density of infected organisms
- large populations: intrinsic (demographic) noise neglected

Exact solution:

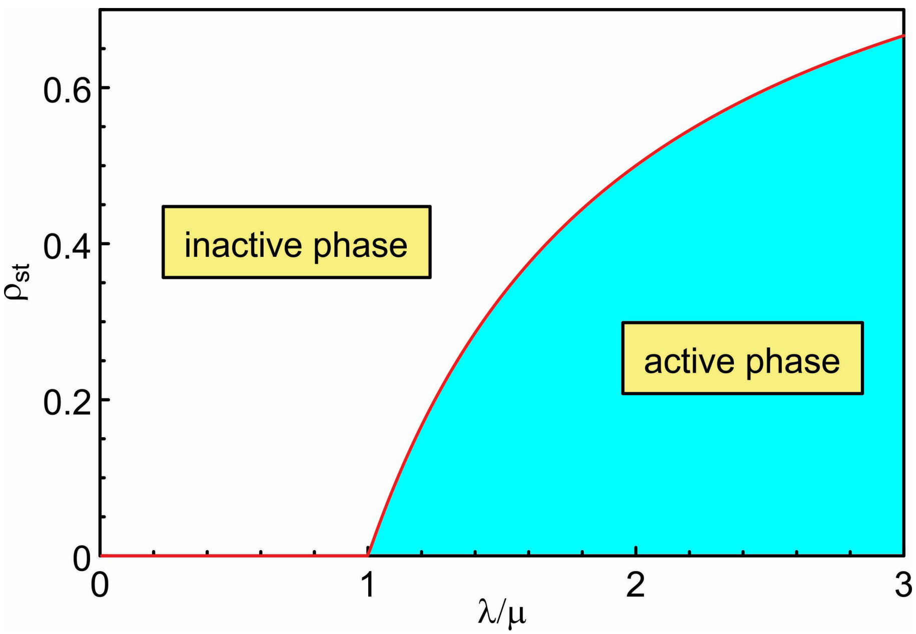
$$\rho^{-1}(t) = a(t)\rho^{-1}(t_0) + c(t)$$

with

$$\begin{aligned} a(t) &= e^{(\mu-\lambda)(t-t_0)} \\ c(t) &= [a(t)-1]\lambda/(\mu-\lambda) \end{aligned}$$



Absorbing state transition



- infection dies out, $\rho(t) \to 0$ for $\mu > \lambda$
- infection survives, $\rho(t) \to \rho_{\rm st} > 0$ for $\mu < \lambda$

$\Rightarrow$ **absorbing state transition** at $\lambda = \mu$

- critical behavior in mean-field **directed percolation** universality class

- stationary density $\rho_{\rm st} = (\lambda - \mu)/\lambda \quad \Rightarrow \quad \beta = 1$
- correlation time $\xi_t = |\lambda - \mu| \quad \Rightarrow \quad \nu_\parallel = z\nu_\perp = 1$
- critical density decay $\rho(t) \sim t^{-1} \quad \Rightarrow \quad \delta = 1$

Criterion $\nu_\parallel > 2$ violated $\Rightarrow$ clean critical point unstable against temporal disorder

Logistic equation with temporal disorder

Temporal disorder (external environmental noise):

- modeled by **time-dependent** healing and infection rates
- temporal disorder is **relevant** perturbation at clean critical point because $\nu_\parallel < 2$

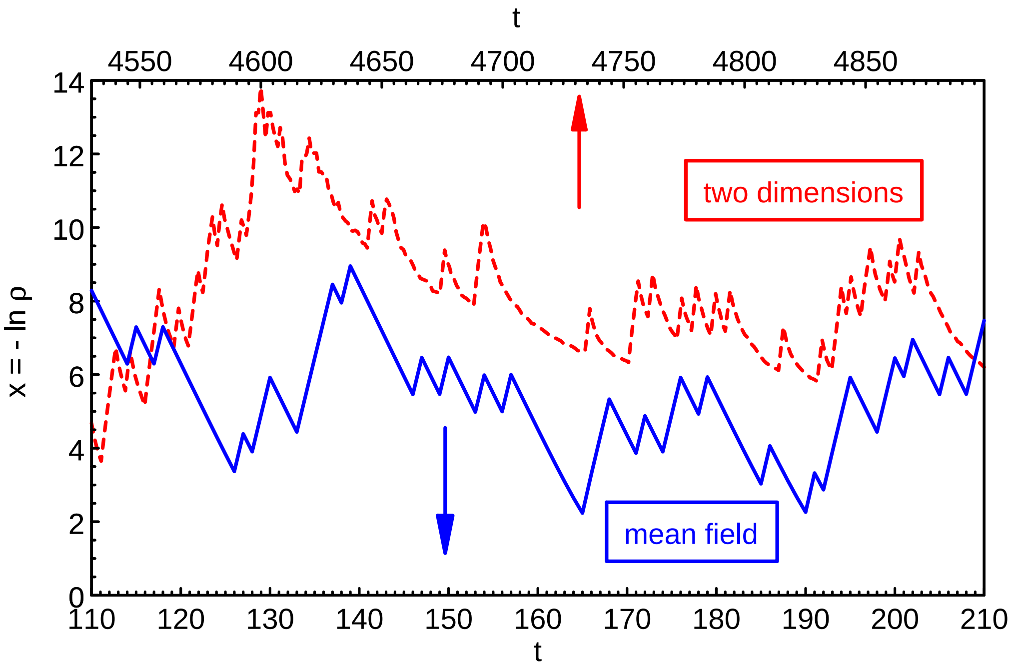
Piecewise constant rates:

- $\mu(t) = \mu_n$, $\lambda(t) = \lambda_n$ in time intervals Δt_n
- closed-form solution in each interval

$\Rightarrow$ linear recurrence for inverse density

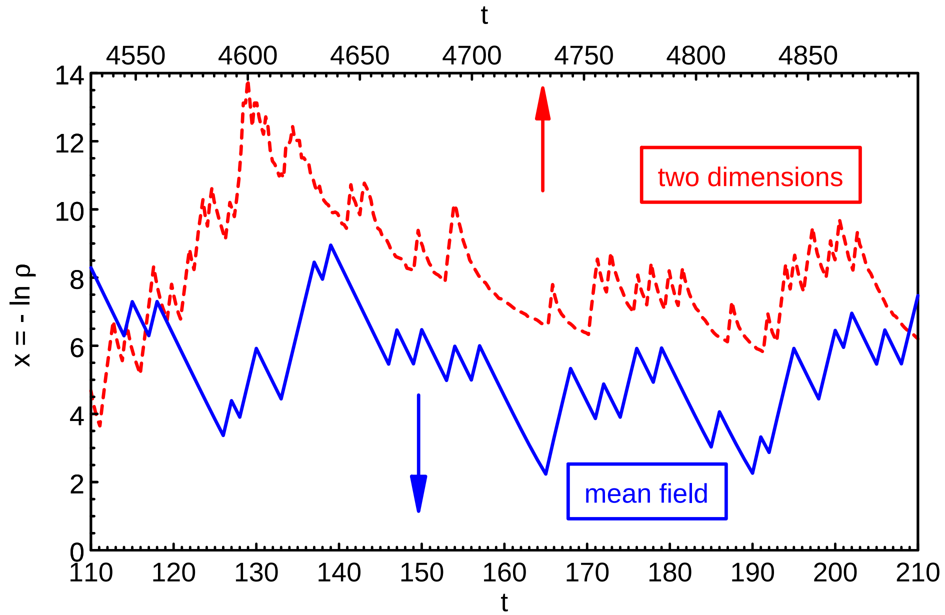
$$\rho_{n+1}^{-1} = a_n \rho_n^{-1} + c_n$$

- multipliers $a_n = \exp[(\mu_n - \lambda_n)\Delta t_n]$
- constants $c_n = (a_n - 1)\lambda_n(\mu_n - \lambda_n)$



Reflected random walk

- recurrence $\rho_{n+1}^{-1} = a_n \rho_n^{-1} + c_n$ maps onto **random walk** for $x_n = -\ln \rho_n$
- effect of constants c_n approximated by **reflecting** boundary condition at $x = 0$
- approximation good if typical $x \gg 1$ (typical $\rho \ll 1$).



Probability distribution $P_n(x)$:

- n time intervals after starting from $\rho_0 = 1$ (i.e., $x_0 = 0$)

$$P_n(x) = \frac{2}{\sqrt{2\pi\sigma^2 n}} e^{-\frac{(x-rn)^2}{2\sigma^2 n}} - \frac{2r}{\sigma^2} e^{\frac{2rx}{\sigma^2}} \Phi\left(\frac{-x - rn}{\sigma n^{1/2}}\right)$$

- $r = \langle \ln a \rangle = \langle (\mu_n - \lambda_n)\, \Delta t_n \rangle$: distance from criticality
- $\sigma^2 = \langle \ln^2 a \rangle - \langle \ln a \rangle^2$

Bulk phases

$$P_n(x) = \frac{2}{\sqrt{2\pi\sigma^2 n}} e^{-\frac{(x-rn)^2}{2\sigma^2 n}} - \frac{2r}{\sigma^2} e^{\frac{2rx}{\sigma^2}} \Phi\left(\frac{-x-rn}{\sigma n^{1/2}}\right)$$

Inactive phase, $r > 0$

- entire distribution $P_n(x)$ shifts to larger x with $n \to \infty$
- almost all disorder realizations have very small density at long times

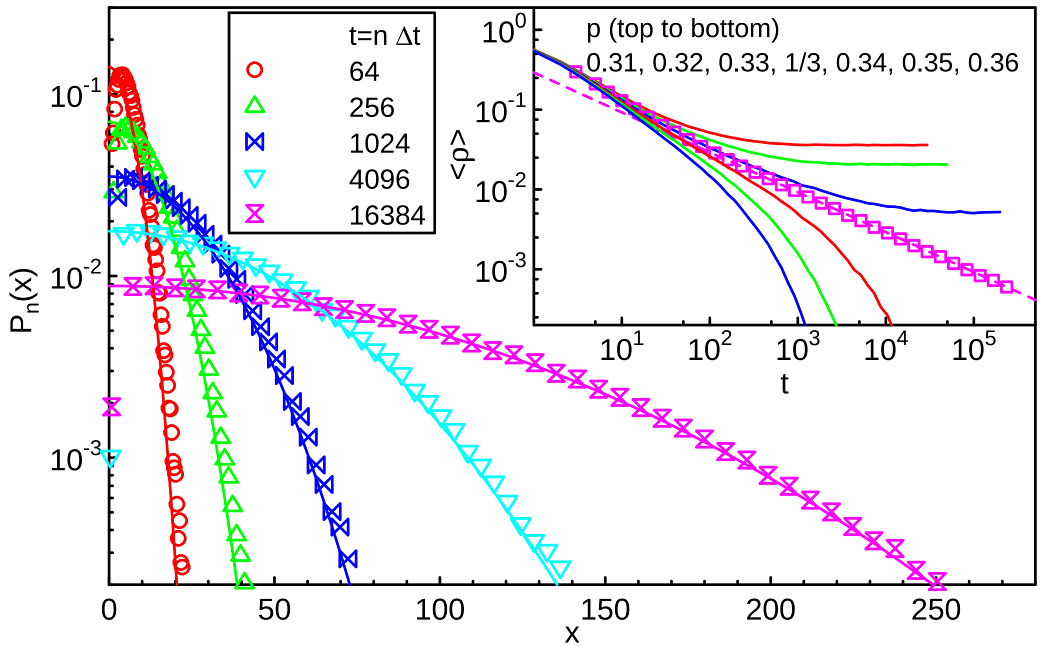
  $\Rightarrow$ **conventional behavior**

Active phase, $r < 0$

- $P_n(x)$ becomes stationary for long times, $P_n(x) \to P_{\rm st}(x) = (2|r|/\sigma^2) e^{-2|r|x/\sigma^2}$

  $\Rightarrow$ **highly singular** density distribution $\bar{P}_{\rm st}(\rho) = (1/\kappa)\rho^{-1+1/\kappa}$
- exponent $\kappa = \sigma^2/(2|r|)$ nonuniversal, diverges at criticality
- **average** density: $\langle \rho_{\rm st} \rangle \sim |r|$
- **typical** density: $\rho_{\rm st}^{\rm typ} \sim \exp(-{\rm const}/|r|)$.

Infinite-noise critical point



At criticality:

$$P_n(x) = 2(2\pi\sigma^2 n)^{-1/2} e^{-x^2/(2\sigma^2 n)}$$

- **broadens without limit** with increasing time
- average density $\langle \rho_n \rangle = 2(2\pi\sigma^2 n)^{-1/2} \sim t^{-1/2}$
- typical density decays much faster $\rho_n^{\mathrm{typ}} = e^{-(2\sigma^2 n/\pi)^{1/2}}$

Infinite-noise critical point is analog of infinite-randomness critical points in spatially disordered systems but with roles of space and time exchanged

Critical exponents:

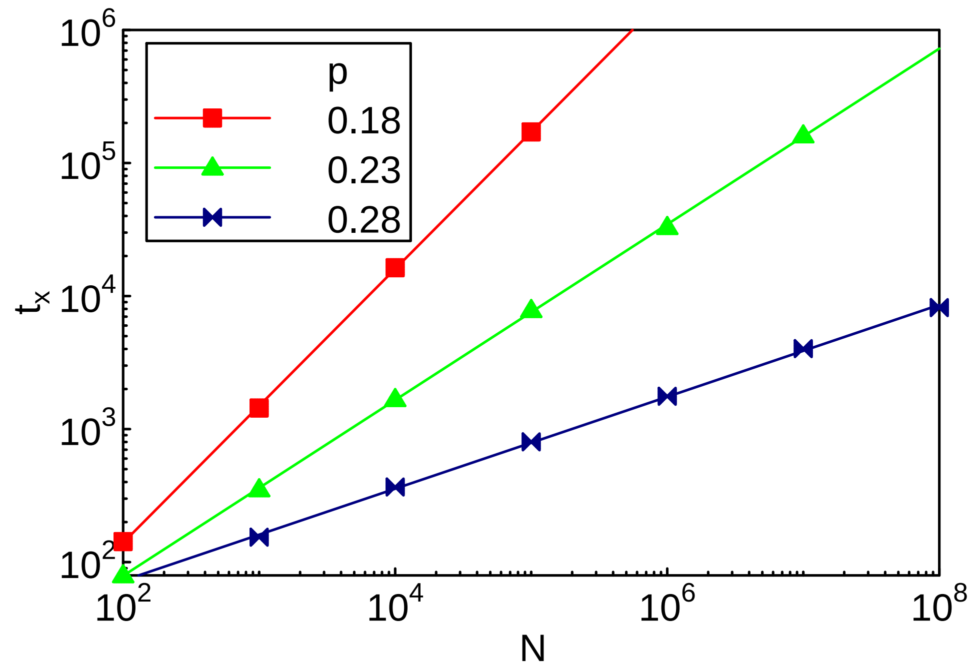
$$\delta = 1/2\ , \quad \beta = 1\ , \quad \nu_\parallel = 2$$

- novel universality class
- $\nu_\parallel$ saturates bound $\nu_\parallel \geq 2$

Temporal Griffiths phase

Life time of finite-size sample:

- **clean** system of size N in active phase: exponential dependence $\tau_N \sim \exp(cN)$
- in presence of **temporal disorder**: power-law dependence $\tau_N \sim N^{1/\kappa}$
- Griffiths exponent $\kappa = \sigma^2/(2|r|)$ diverges at criticality, same exponent as in density distribution



Temporal Griffiths phase:

- region of the **active** phase that features anomalous power-law dependence of the life time on the system size (F. Vazquez et al., PRL, **106**, 235702 (2011))

- Stability criterion
- Logistic evolution equation
- **Strong-noise renormalization group**
- Contact process
- Monte-Carlo simulations

Real-time (strong-noise) renormalization group

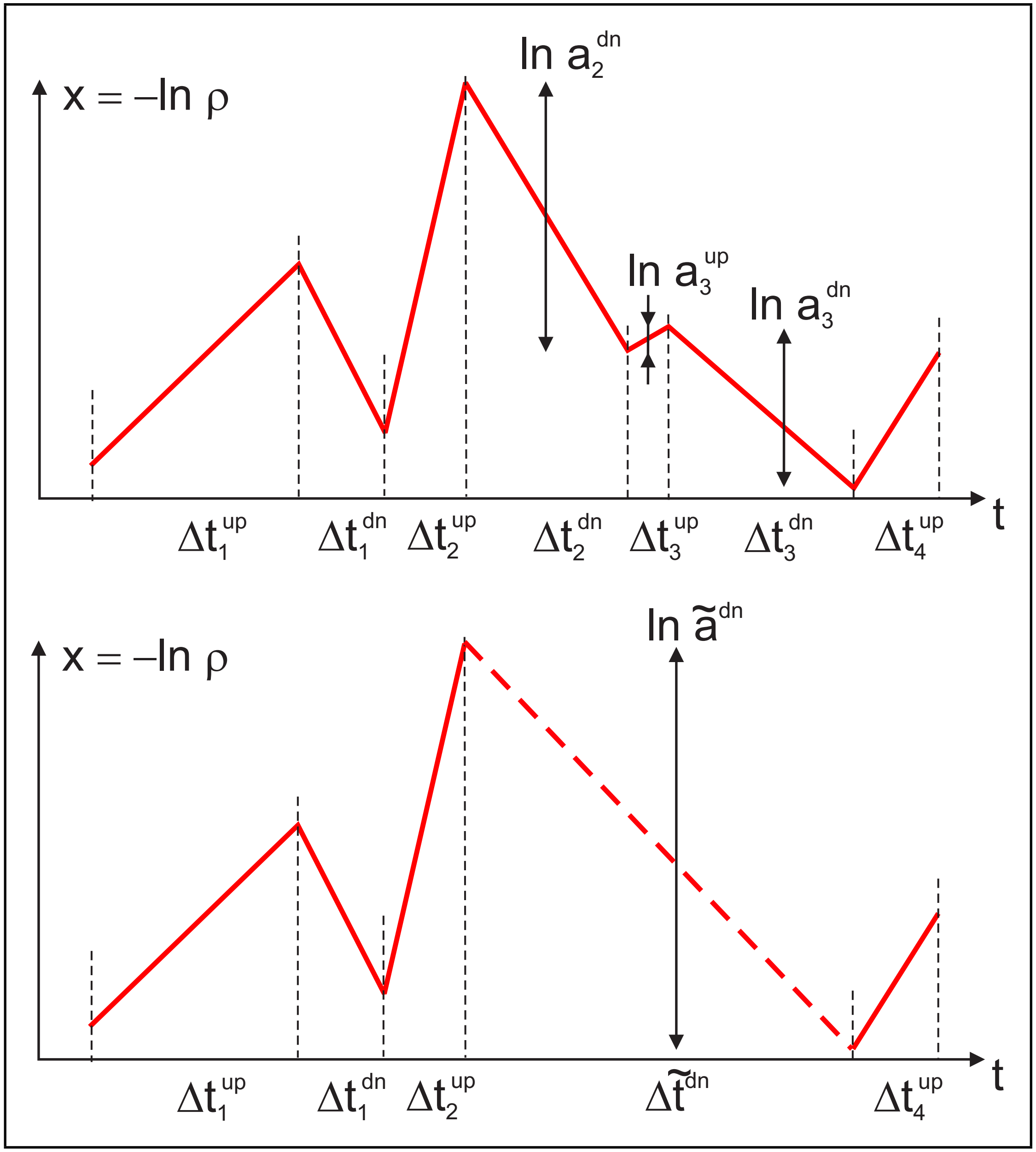
Basic idea:

iteratively eliminate smallest up or down segment by combining it with its neighbors, $\Omega = \min(a_i^{\rm up}, 1/a_i^{\rm dn})$

RG recursions:

$$\begin{aligned}
\tilde{a}^{\rm up} &= a_{i+1}^{\rm up} a_i^{\rm up}/\Omega \\
1/\tilde{a}^{\rm dn} &= (1/a_i^{\rm dn})\,(1/a_{i-1}^{\rm dn})/\Omega \\
\Delta\tilde{t}^{\rm up} &= \Delta t_i^{\rm up} + \Delta t_i^{\rm dn} + \Delta t_{i+1}^{\rm up} \\
\Delta\tilde{t}^{\rm dn} &= \Delta t_{i-1}^{\rm dn} + \Delta t_i^{\rm up} + \Delta t_i^{\rm dn}
\end{aligned}$$

$$\tilde{c}^{\rm up} = a_{i+1}^{\rm up} a_i^{\rm dn} c_i^{\rm up} + a_{i+1}^{\rm up} c_i^{\rm dn} + c_{i+1}^{\rm up}$$
$$\tilde{c}^{\rm dn} = a_i^{\rm dn} a_i^{\rm up} c_{i-1}^{\rm dn} + a_i^{\rm dn} c_i^{\rm up} + c_i^{\rm dn}$$



Solving the strong-noise RG

- recursions for multipliers and time intervals equivalent to Fishers's RG for RTIM
- iterate RG step: flow equations for **distributions $\mathcal{R}$ and $\mathcal{P}$** of a_i^{up} and a_i^{dn} (using logarithmic variables $\Gamma = \ln \Omega$, $\beta = \ln a^{\mathrm{up}} - \Gamma$, and $\zeta = -\ln a^{\mathrm{dn}} - \Gamma$)

$$\frac{\partial \mathcal{R}}{\partial \Gamma} = \frac{\partial \mathcal{R}}{\partial \beta} + (\mathcal{R}_0 - \mathcal{P}_0)\mathcal{R} + \mathcal{P}_0 \left(\mathcal{R} \overset{\beta}{\otimes} \mathcal{R} \right)$$

$$\frac{\partial \mathcal{P}}{\partial \Gamma} = \frac{\partial \mathcal{P}}{\partial \zeta} + (\mathcal{P}_0 - \mathcal{R}_0)\mathcal{P} + \mathcal{R}_0 \left(\mathcal{P} \overset{\zeta}{\otimes} \mathcal{P} \right)$$

- can be solved via ansatz $\mathcal{R}(\beta;\Gamma) = \mathcal{R}_0 e^{-\mathcal{R}_0 \beta}$, $\mathcal{P}(\zeta;\Gamma) = \mathcal{P}_0 e^{-\mathcal{P}_0 \zeta}$
- flow equations reduce to ordinary differential equations

$$d\mathcal{R}_0/d\Gamma = -\mathcal{R}_0 \mathcal{P}_0 \ , \quad d\mathcal{P}_0/d\Gamma = -\mathcal{R}_0 \mathcal{P}_0$$

- fixed points can be found analytically

$\Rightarrow$ critical point is **analogous to random transverse-field Ising chain**, but with **time playing role of length**

Renormalization group results

Fixed points:

- critical point: $\mathcal{R}_0 = \mathcal{P}_0 = 1/\Gamma$, time scale $\Delta\tilde{t} \sim \Gamma^2$,
- inactive phase: $\mathcal{R}_0 = \exp(-\Gamma/\kappa)/\kappa$, $\mathcal{P}_0 = 1/\kappa$ with constant κ, time scale $\Delta\tilde{t} \sim \exp(\Gamma/\kappa)$
- active phase: $\mathcal{R}_0 = 1/\kappa$, $\mathcal{P}_0 = \exp(-\Gamma/\kappa)/\kappa$, $\Delta\tilde{t} \sim \exp(\Gamma/\kappa)$

Observables:

- lifetime τ_N: run RG until the typical upward multiplier a^{up} reaches N
  τ_N is typical time interval at this RG scale

$$\tau_N \sim \begin{cases} N^{1/\kappa} & \text{active phase} \\ (\ln N)^2 & \text{criticality} \\ \ln N & \text{inactive phase} \end{cases}$$

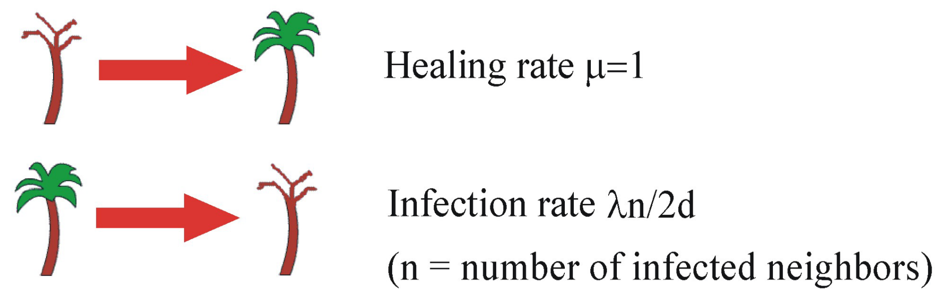
- width of $P(x)$ at criticality: proportional to $\ln a^{\mathrm{up}} \sim \Gamma \sim t^{1/2}$

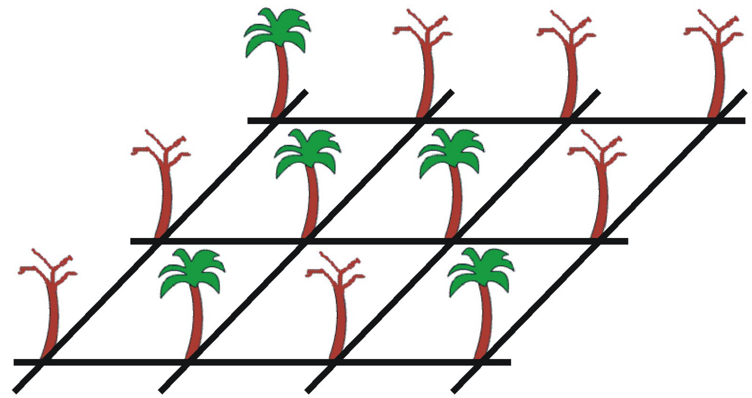
Strong-noise RG reproduces results of the reflected-random-walk approach

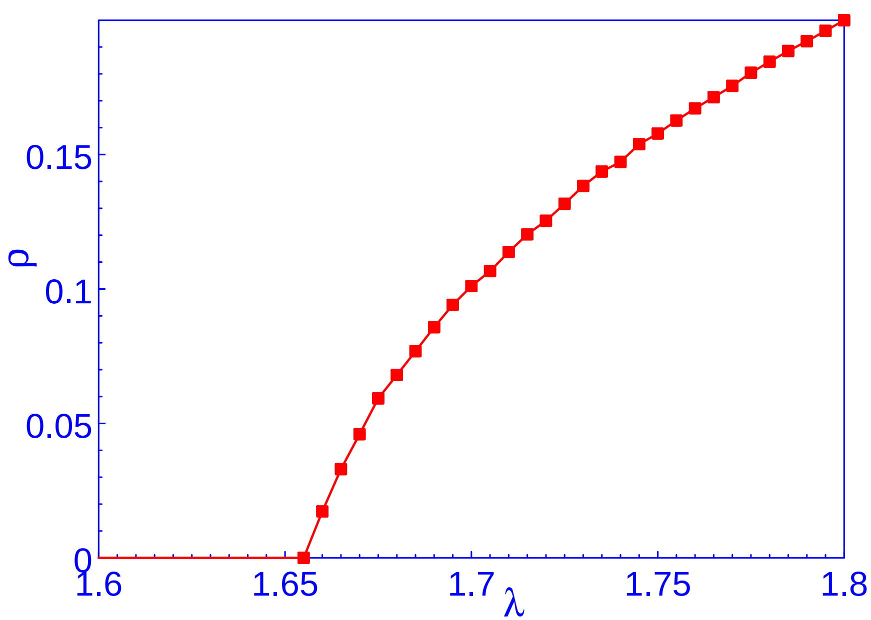
- Stability criterion
- Logistic evolution equation
- Strong-noise renormalization group
- **Contact process (in finite dimensions)**
- Monte-Carlo simulations

Contact process

- d**-dimensional hypercubic lattice**
- each site can be either healthy (inactive) or sick (active)
- sick sites heal with rate μ
- healthy sites get infected by neighbors with rate λ







- absorbing state transition in **directed percolation** universality class
- correlation time exponent violates Kinzel's criterion $\nu_\parallel > 2$ in all d

$\Rightarrow$ temporal disorder is **relevant** perturbation

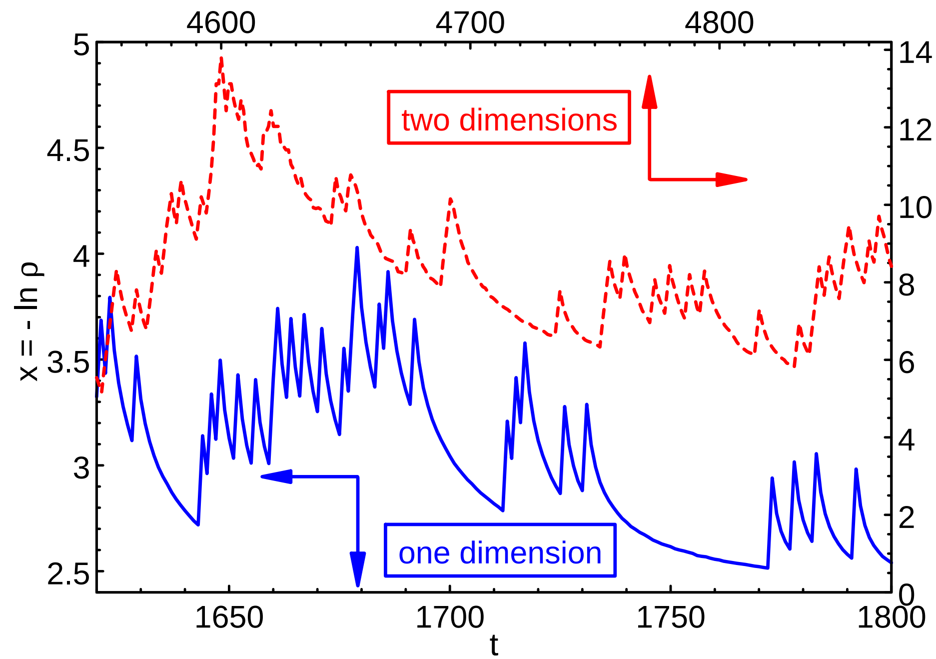
Strong-noise RG in finite dimensions

Strong disorder:

- individual spreading and decay segments are deep in bulk phases

$\Rightarrow$ neglect spatial fluctuations

$\Rightarrow$ develop theory in terms of $\rho(t)$ only



Renormalization group recursions:

- density decay is **exponential** in time (as in mean-field case)

  recursion for decay segments multiplicative

- spreading is **ballistic**, $\rho(t) \approx \rho_0(1+bt)^d$, rather than exponential

  additive rather than multiplicative RG recursion

$$\begin{aligned} \tilde{a}^{\rm up} &= a^{\rm up}_{i+1} a^{\rm up}_i / \Omega \\ (1/\tilde{a}^{\rm dn})^{1/d} &= (1/a^{\rm dn}_i)^{1/d} + (1/a^{\rm dn}_{i-1})^{1/d} - \Omega^{1/d} \end{aligned}$$

Flow equations

- reduced variables $\beta = \ln a^{\rm up} - \Gamma$ and $\zeta = d[(\Omega a^{\rm dn})^{-1/d} - 1]$
- flow equations for **distributions $\mathcal{R}$ and $\mathcal{P}$** of $a_i^{\rm up}$ and $a_i^{\rm dn}$

$$
\begin{aligned}
\frac{\partial \mathcal{R}}{\partial \Gamma} &= \frac{\partial \mathcal{R}}{\partial \beta} + (\mathcal{R}_0 - \mathcal{P}_0)\mathcal{R} + \mathcal{P}_0 \left(\mathcal{R} \overset{\beta}{\otimes} \mathcal{R} \right) \\
\frac{\partial \mathcal{P}}{\partial \Gamma} &= \left(1 + \frac{\zeta}{d}\right) \frac{\partial \mathcal{P}}{\partial \zeta} + \left(\mathcal{P}_0 - \mathcal{R}_0 + \frac{1}{d} \right) \mathcal{P} + \mathcal{R}_0 \left(\mathcal{P} \overset{\zeta}{\otimes} \mathcal{P} \right) \ .
\end{aligned}
$$

- solved by exponential ansatz $\mathcal{R}(\beta;\Gamma) = \mathcal{R}_0 e^{-\mathcal{R}_0 \beta}$, $\mathcal{P}(\zeta;\Gamma) = \mathcal{P}_0 e^{-\mathcal{P}_0 \zeta}$

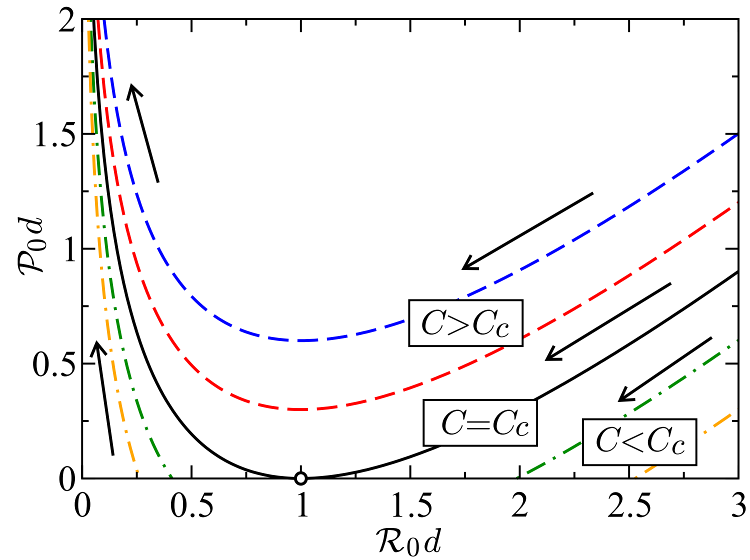
$$
d\mathcal{R}_0/d\Gamma = -\mathcal{R}_0 \mathcal{P}_0 \ , \quad d\mathcal{P}_0/d\Gamma = (1/d - \mathcal{R}_0)\mathcal{P}_0
$$

$\Rightarrow$ **Kosterlitz-Thouless type flow equations** (mean-field case recovered for $d \to \infty$)

Infinite-noise critical point

Fixed points:

- line of fixed points at $\mathcal{P}_0^* = 0, \mathcal{R}_0^*$ arbitrary
- FPs stable for $\mathcal{R}_0 > 1/d$ (active phase)
- FPs unstable for $\mathcal{R}_0 < 1/d$, flow towards $\mathcal{R}_0 = 0$, $\mathcal{P}_0 = \infty$ (inactive phase)



Critical fixed point:

- endpoint $\mathcal{P}_0 = 0$, $\mathcal{R}_0 = 1/d$ of active FP line
- distribution $P(x)$ of $x = -\ln\rho$ **broadens** with time but only **logarithmically**

$\Rightarrow$ **infinite-noise critical point**, but in **different** universality class than mean-field case

- Griffiths singularities in the life time of finite-size systems, $\tau_N \sim N^{1/\kappa}$
- Griffiths exponent κ saturates at $\kappa_c = d$

Critical behavior:

$$\langle\rho\rangle \sim (\ln t)^{-\bar\delta} \quad \text{with} \quad \bar\delta = 1$$

$$\rho_{\text{st}} \sim |r|^{\beta} \quad \text{with} \quad \beta = 1/2$$

$$\xi_t \sim \xi^z \quad \text{with} \quad z = 1$$

$$\ln \xi_t \sim |r|^{-\bar\nu_\parallel} \quad \text{with} \quad \bar\nu_\parallel = 1/2$$

the usual correlation time exponent $\nu_\parallel$ is formally infinite

Scaling theory

- **density of active sites:**

$$\rho_{\rm av}(r,t,L) = (\ln b)^{-\beta/\bar\nu_\perp} \rho_{\rm av}(r(\ln b)^{1/\bar\nu_\perp}, tb^{-z}, Lb^{-1})$$

- **survival probability** (starting from single seed site)

$$P_s(r,t,L) = (\ln b)^{-\beta/\bar\nu_\perp} P_s(r(\ln b)^{1/\bar\nu_\perp}, tb^{-z}, Lb^{-1})$$

- **radius** of spreading cluster and its **number of sites**

$$\begin{aligned} R(r,t,L) &= b(\ln b)^{-y_R} R(r(\ln b)^{1/\bar\nu_\perp}, tb^{-z}, Lb^{-1}) \\ N_s(r,t,L) &= b^d(\ln b)^{-y_N} N_s(r(\ln b)^{1/\bar\nu_\perp}, tb^{-z}, Lb^{-1}) \end{aligned}$$

Exponents:

$\beta = 1$, $\bar\nu_\perp = 1/2$, $z = 1$

Time dependencies at criticality:

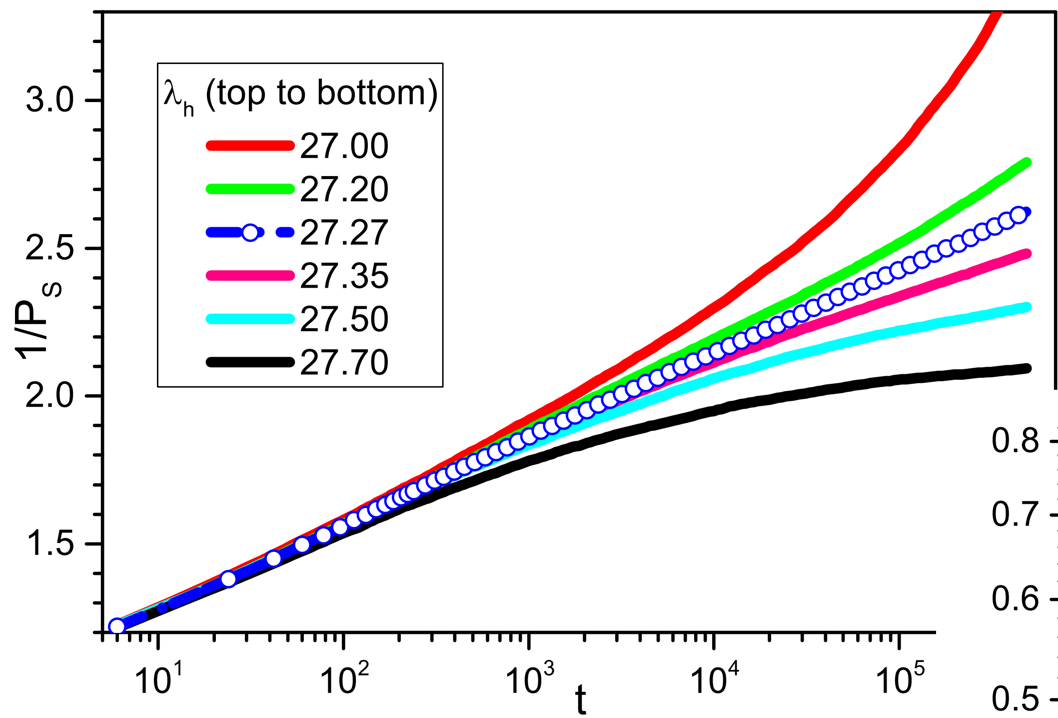
$$\begin{aligned} \rho_{\rm av}(t) &\sim (\ln t)^{-1} \\ P_s(t) &\sim (\ln t)^{-1} \\ R(t) &\sim t(\ln t)^{-y_R} \\ N_s(t) &\sim t^d(\ln t)^{-y_N} \end{aligned}$$

- Stability criterion
- Logistic evolution equation
- Strong-noise renormalization group
- Contact process
- **Monte-Carlo simulations**

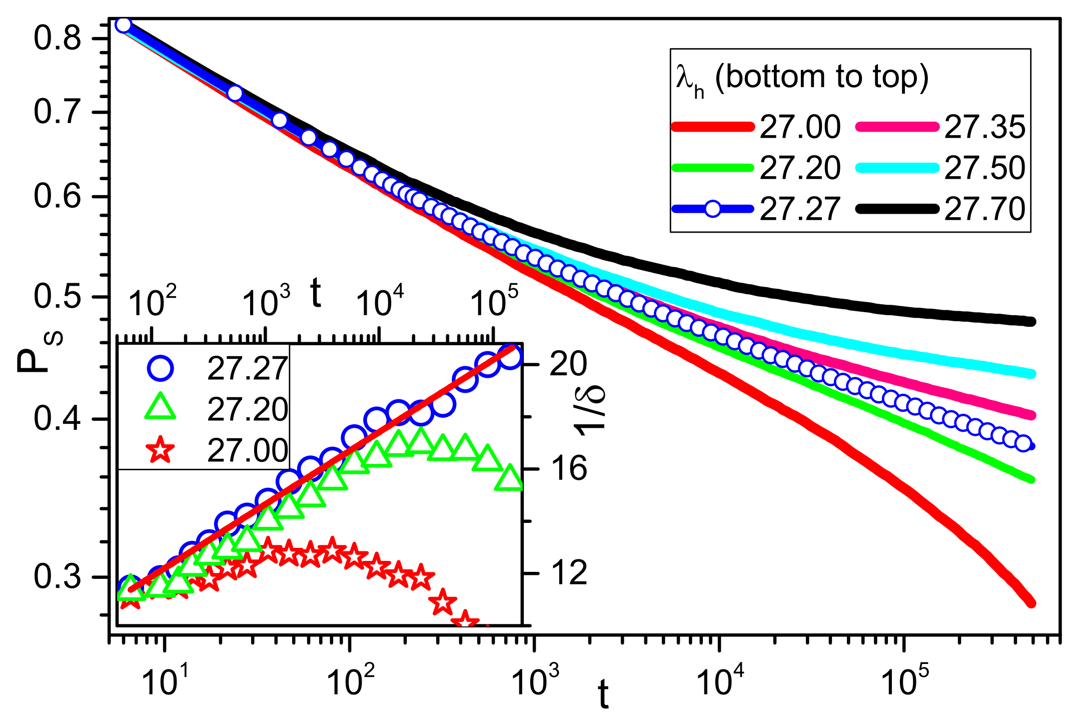
Spreading simulations in 1d

Strong disorder:

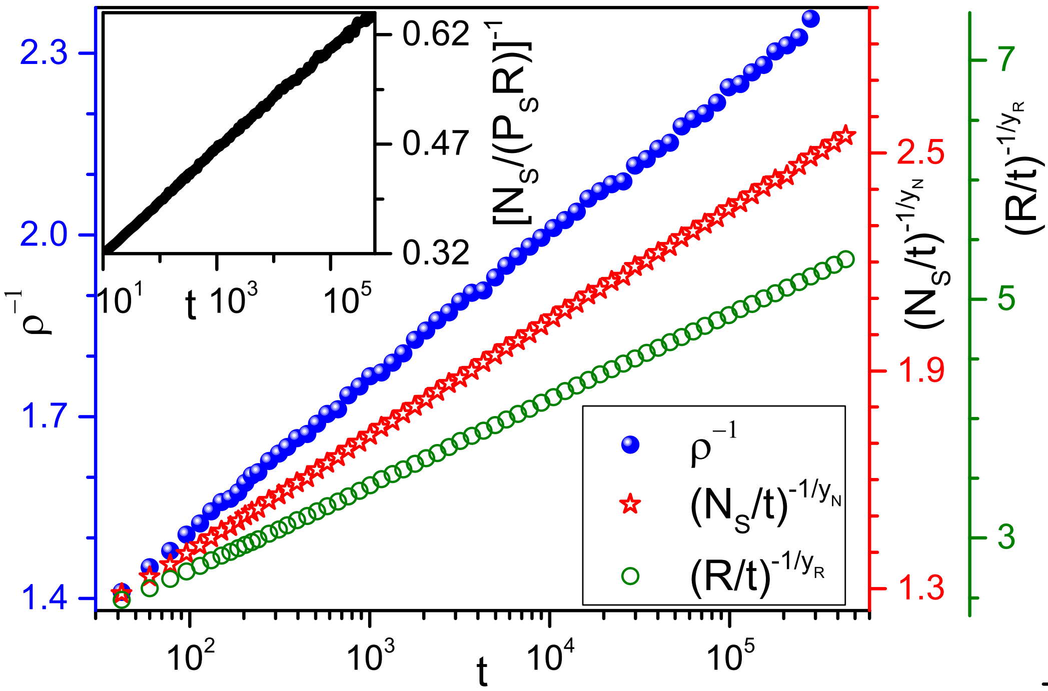
$W_\lambda(\lambda) = p\,\delta(\lambda - \lambda_h) + (1-p)\,\delta(\lambda - \lambda_h/20)$ with $p = 0.8$ and time interval $\Delta t = 6$.



Survival Probability P_s

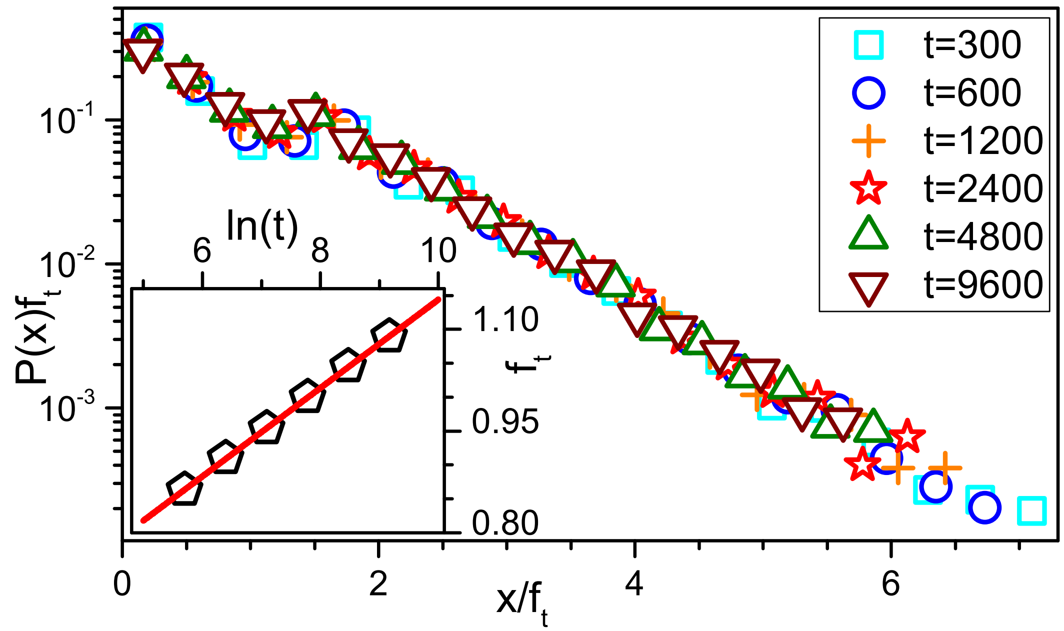


Spreading simulations in 1d

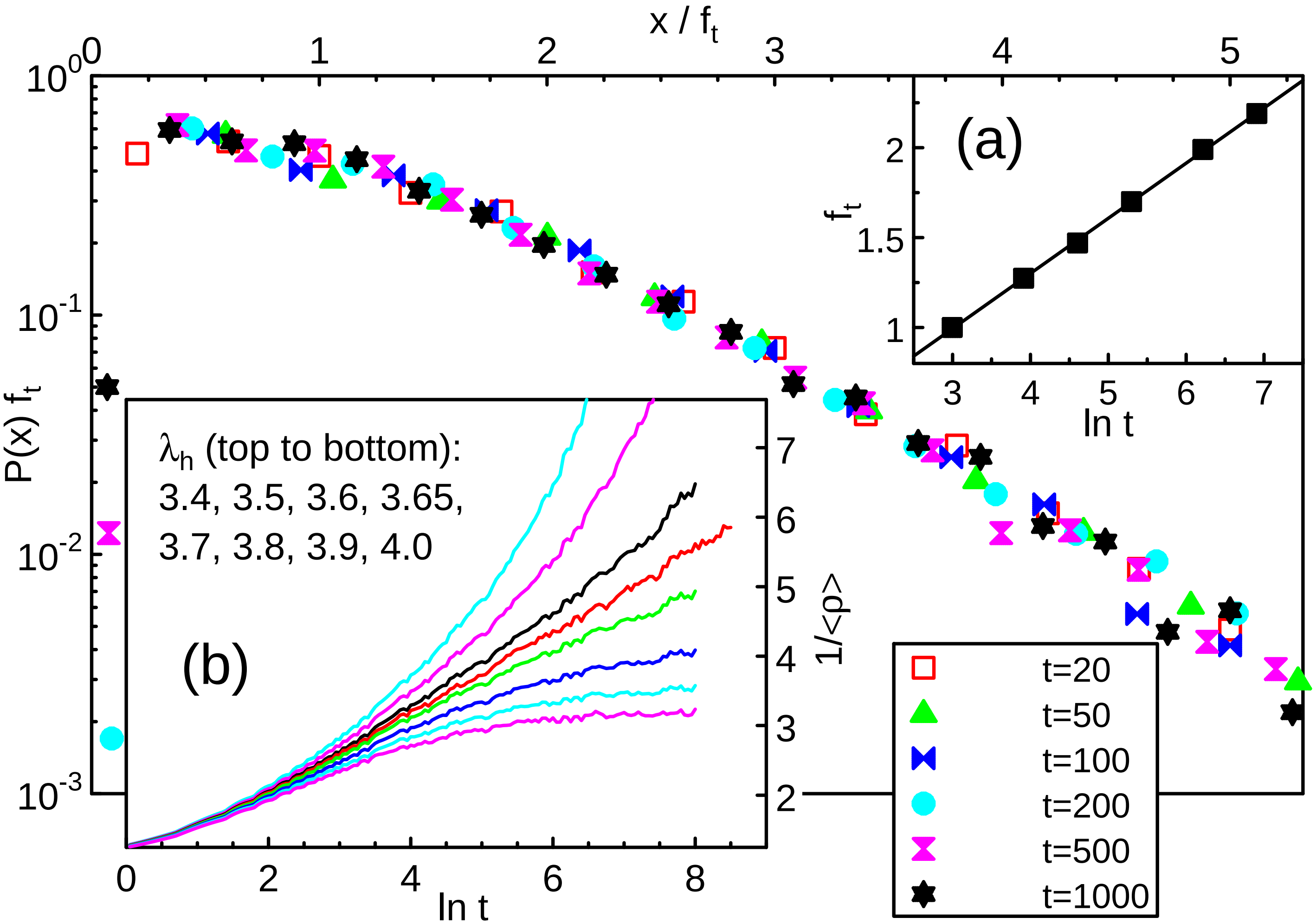


Radius and size of spreading cluster

Probability distribution $P(x)$
$(x = -\ln \rho)$



Density decay simulations in 2D



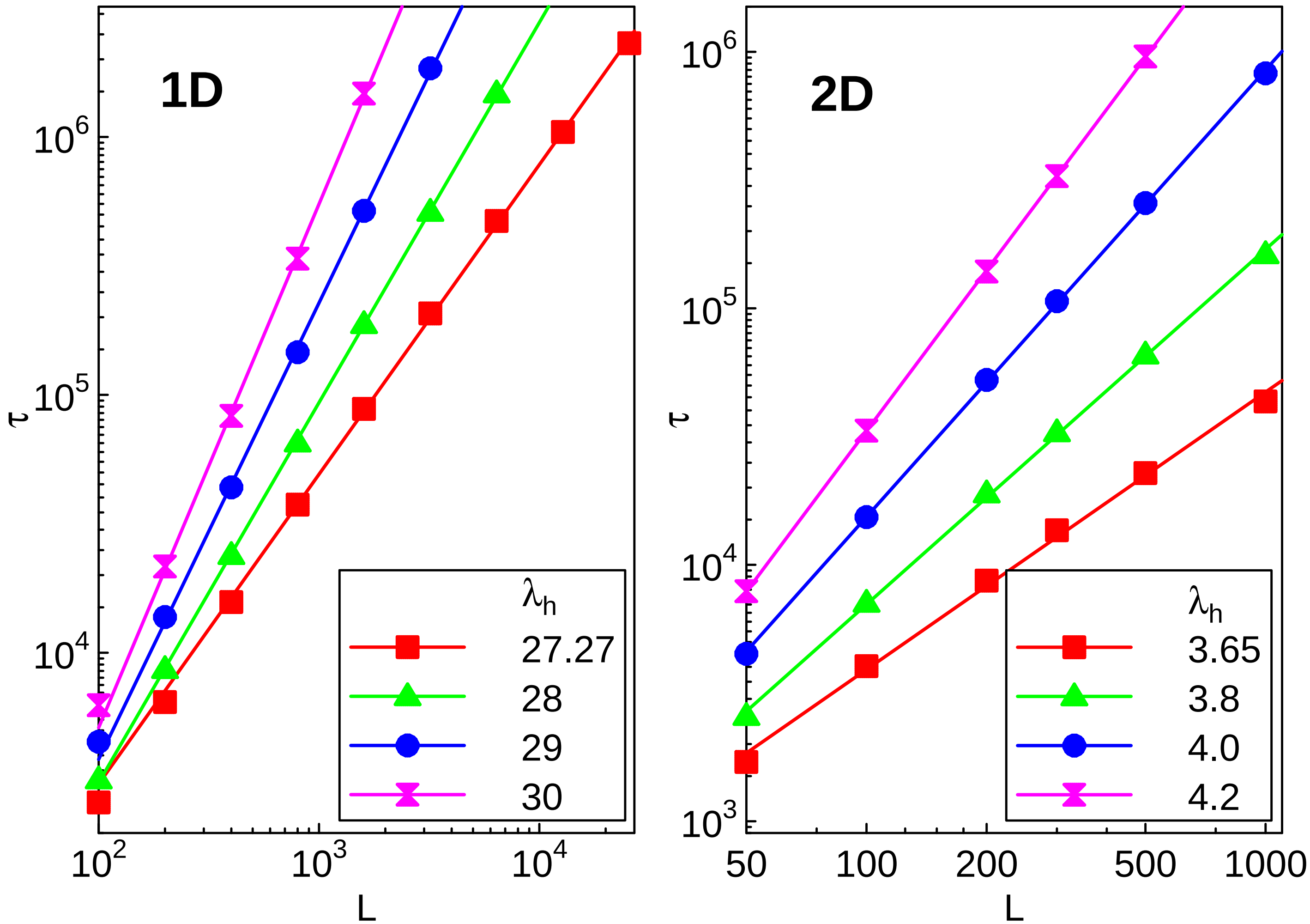
Main: $P(x)$ at criticality for different times t, scaled such that the curves coincide.

Inset (a): Scale factor f_t vs. $\ln t$, confirming the logarithmic broadening of $P(x)$

Inset (b): average density $\langle \rho(t) \rangle$ for varying infection rate λ

Temporal Griffiths phase

Life time of finite-size samples:



at criticality: $\kappa_c \approx 0.85$ (in 1d) and 1.9 (in 2d)

Conclusions

- random environmental noise (temporal disorder) destabilizes clean absorbing state transitions
- problem can be studied using real-time **"strong noise" renormalization group**
- leads to novel, exotic **"infinite-noise" critical points** at which the effective noise amplitude diverges on long time scales
- implies enormous density fluctuations, density distribution at criticality becomes **infinitely broad** (even on a logarithmic scale)
- renormalization group theory confirmed by numerical simulation for both the mean-field case and finite dimensions

Infinite-noise critical points in temporally disordered systems are the analogs of infinite-randomness critical points in spatially disordered systems, but with the roles of space and time exchanged.

For more details see: Europhys. Lett. **112**, 30002 (2015), arXiv:1507.05677 and arXiv:1603.08075