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OSCILLATION CRITERIA FOR FOURTH ORDER NONLINEAR POSITIVE DELAY DIFFERENTIAL EQUATIONS WITH A MIDDLE TERM

SAID GRACE AND ELVAN AKIN

Department of Engineering Mathematics, Cairo University,
Orman, Giza 12221, Egypt
Missouri University of Science Technology,
310 Rolla Building, MO, 65409-0020

ABSTRACT. In this article, we establish some new criteria for the oscillation of fourth order nonlinear delay differential equations of the form

$$x^{(4)}(t) + p(t)x^{(2)}(t) + q(t)f(x(g(t))) = 0$$

provided that the second order equation

$$z^{(2)}(t) + p(t)z(t) = 0$$

is nonoscillatiory or oscillatory. This equation with $g(t) = t$ is considered in [8] and some oscillation criteria for this equation via certain energy functions are established. Here, we continue the study on the oscillatory behavior of this equation via some inequalities.

Key words. oscillation, differential equations, higher order, delay.

AMS (MOS) Subject Classification. 34C10, 39A10.

1. INTRODUCTION

In this article, we consider nonlinear fourth order functional differential equations of the form

$$x^{(4)}(t) + p(t)x^{(2)}(t) + q(t)f(x(g(t))) = 0, \quad t \geq t_0 > 0 \tag{1.1}$$

together with the associated second order equation

$$z^{(2)}(t) + p(t)z(t) = 0. \tag{1.2}$$

We assume that

1. $p, q \in C([t_0, \infty), \mathbb{R}^+)$;
2. $g \in C^1([t_0, \infty), \mathbb{R}^+)$ such that $g(t) < t$, $g'(t) \geq 0$ and $\lim_{t\to\infty} g(t) = \infty$;
3. $f \in C(\mathbb{R}, \mathbb{R})$ such that $xf(x) > 0$ and $\dfrac{f(x)}{x^\beta} \geq k > 0$ for $x \neq 0$, where k is a constant and β is the ratio of positive odd integers.

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We restrict our attention to those solutions of equation (1.1) which exist on $I = [t_0, \infty)$ and satisfy the condition

$$\sup\{|x(t)| : t_1 \le t < \infty\} > 0 \text{ for } t_1 \in [t_0, \infty).$$

Such a solution is called *oscillatory* if it has arbitrarily large zeros, otherwise it is called *nonoscillatory*. Equation (1.1) is said to be oscillatory if it has an oscillatory solution. The oscillatory behavior of fourth order differential equations with middle term enjoys a great deal of interest, see [1]-[4] and [6]-[17] references contained therein. The important role in the investigation of equation (1.1) is played by the fact whether the associated second order linear equation (1.2) is oscillatory or nonoscillatory.

In [8], they considered (1.1) with $g(t) = t$ and employed an approach based on a suitable energy function for equation (1.1) and a comparison method for equation (1.1) and obtained the following result, see [[8], Theorem 3.1].

Theorem 1.1. *Assume that $\beta = 1$, equation (1.2) is nonoscillatory,*

$$\lim_{t\to\infty} \frac{q(t)}{p(t)} = \infty, \quad p^2(t) \le 4q(t) \text{ for all large } t$$

and

$$\int^{\infty} s^2 q(s) ds = \infty.$$

Then (1.1) with $g(t) = t$ is oscillatory.

If $\beta < 1$ and equation (1.2) is oscillatory, the following oscillation criterion for equation (1.1) has been proved in [8, Theorem 3.4].

Theorem 1.2. *Let $\beta < 1$ and equation (1.2) be oscillatory. Assume that $p(t) \ge p > 0$, $p'(t) \le 0$ and $p''(t) > 0$ and*

$$\lim_{t\to\infty} t^{2(\beta-1)} q(t) = \infty.$$

Then (1.1) with $g(t) = t$ is oscillatory.

Motivated by these results in [8] which are applicable to equation (1.1) with $g(t) = t$, we study the oscillation of equation (1.1) with delay. We allow that the function p can tend to a real number or to infinity as $t \to \infty$ and both cases that the corresponding second order equation (1.2) is nonoscillatory (oscillatory) are considered.

2. MAIN RESULTS

To obtain our results, we need the following lemmas.

Lemma 2.1 ([1, 2])**.** *Every eventually positive solution $x(t)$ of equation (1.1) is one of the following types:*

- *Type* (a). $x(t) > 0$, $x'(t) > 0$ *and* $x^{(2)}(t) < 0$ *for large* t,
- *Type* (b). $x(t) > 0$, $x'(t) > 0$, $x^{(2)}(t) > 0$ *and* $x^{(3)}(t) > 0$ *for large* t,
- *Type* (c). $x^{(2)}(t)$ *changes sign eventually.*

Moreover, if equation (1.2) *is nonoscillatory, then* x *is of Type* (a) *or Type* (b) *and if equation* (1.2) *is oscillatory, then* x *is of Type* (a) *or Type* (c).

Lemma 2.2. *Let* $\beta \leq 1$ *and equation* (1.2) *be nonoscillatory. If*

$$\int^{\infty} (p(s) + g^{2\beta}(s)q(s))ds = \infty, \tag{2.1}$$

then equation (1.1) *has no solution of Type* (b), *i.e., every eventually positive solution of* (1.1) *is of Type* (a).

Proof. Let x be an eventually positive solution of equation (1.1) of Type (b). There exist two positive constants c_1 and c_2 and $t_1 \geq t_0$ such that $x^{(2)}(t) \geq c_1$ and so we get $x(g(t)) \geq c_2 g^2(t)$ for all $t \geq t_1$. Integrating equation (1.1) from t_1 to t, we have

$$\begin{aligned} \infty &> -x^{(3)}(t) + x^{(3)}(t_1) \\ &\geq \int_{t_1}^{t} (c_1 p(s) + k c_2^{\beta} g^{2\beta}(s) q(s)) ds \\ &\geq C \int_{t_1}^{t} (p(s) + g^{2\beta}(s)q(s))ds \to \infty \text{ as } t \to \infty, \end{aligned}$$

where $C = \max\{c_1, kc_2^{\beta}\}$ which contradicts the fact that $x^{(3)}(t)$ is bounded. This completes the proof. □

Lemma 2.3. *Let* $\beta \leq 1$ *and equation* (1.2) *be nonoscillatory. If for every positive constant* c, *the first order delay equation*

$$y'(t) + ckq(t)g^{3\beta}(t)y^{\beta}(g(t)) = 0, \tag{2.2}$$

is oscillatory, then equation (1.1) *has no solution of Type* (b), *i.e., every eventually positive solution of* (1.1) *is of Type* (a).

Proof. Let x be an eventually positive solution of equation (1.1) of Type (b). It is easy to see that there exist a constant c^*, $0 < c^* < 1$ and $t_1 \geq t_0$ such that

$$x^{(2)}(t) \geq c^* x^{(3)}(t) \text{ for } t \geq t_1. \tag{2.3}$$

Integrating (2.3) twice from t_1 to t, we see that there exist a constant $c > 0$ and a $t_2 \geq t_1$ such that

$$x(t) \geq ct^3 x^{(3)}(t) \text{ for } t \geq t_2. \tag{2.4}$$

Using the inequalities (2.3) and (2.4) in equation (1.1), we get

$$y'(t) + c^* p(t) t y(t) + kc^{\beta} q(t) g^{3\beta}(t) y^{\beta}(g(t)) \leq 0$$

or

$$y'(t) + kc^{\beta}q(t)g^{3\beta}(t)y^{\beta}(g(t)) \le 0,$$

where $y(t) = x^{(3)}(t) > 0$ for $t \ge t_2$. It follows from Theorem 1 in [3] that the corresponding equation (2.2) also has a positive solution. This gives us a contradiction. □

The following corollary is an immediate consequence of Lemma 2.3.

Corollary 2.4. *Let $\beta \le 1$ and equation* (1.2) *be nonoscillatory. If for every positive constant c,*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} q(s)g^{3\beta}(s)ds > \frac{1}{c^{\beta}ke}, \tag{2.5}$$

then equation (1.1) *has no solution of Type* (b).

Lemma 2.5. *Let $\beta \le 1$ and equation* (1.2) *be (non)oscillatory. If there exist a function $h \in C^1(I, \mathbb{R})$ such that $g(t) \le h(t) \le t$, $h'(t) \ge 0$ for $t \ge t_0$ such that the second order inequality*

$$w''(t) \ge P(t)w(h(t)), \tag{2.6}$$

where $P(t) = cq(t)g^{\beta}(t)(h(t) - g(t))^{\beta} - p(t) > 0$ for some constant $c > 0$, has no positive bounded solutions, then equation (1.1) *has no solution of Type* (a).

Proof. Let x be an eventually positive solution of equation (1.1) of Type (a). It is easy to see that there exist a constant c^* such that $0 < c^* < 1$ and $t_1 \ge t_0$ such that

$$x(t) \ge c^* t x'(t) \text{ for } t \ge t_1. \tag{2.7}$$

Using (2.7) in equation (1.1), one can easily find that

$$y^{(3)}(t) + p(t)y'(t) + (c^*)^{\beta}kq(t)g^{\beta}(t)y^{\beta}(g(t)) \le 0 \text{ for } t \ge t_1, \tag{2.8}$$

where $y(t) = x'(t)$. Clearly, we see that $y(t) > 0$, $y'(t) < 0$ and $y''(t) > 0$ for $t \ge t_1$. Now for $v \ge u \ge t_1$ we have

$$y(u) \ge y(u) - y(v) = -\int_u^v y'(s)ds \ge (v - u)(-y'(v)). \tag{2.9}$$

For $t \ge t_1$ setting $u = g(t)$ and $v = h(t)$ in (2.9), we get

$$y(g(t)) \ge (h(t) - g(t))(-y'(h(t)))). \tag{2.10}$$

Using (2.10) in (2.8), we get

$$\begin{aligned} w''(t) + p(t)w(t) &\ge k(c^*)^{\beta}q(t)g^{\beta}(t)(h(t) - g(t))^{\beta}w^{\beta}(h(t)) && (2.11)\\ &= k(c^*)^{\beta}q(t)g^{\beta}(t)(h(t) - g(t))^{\beta}w^{\beta-1}(h(t))w(h(t)), && (2.12)\end{aligned}$$

where $w(t) = -y'(t) > 0$ for $t \ge t_1$. Using the fact that $g(t) \le h(t) \le t$, $\beta \le 1$ and $w(t)$ is decreasing, we obtain

$$w''(t) + p(t)w(h(t)) \ge (c^*)^{\beta}Cq(t)g^{\beta}(t)\left(h(t) - g(t)\right)^{\beta} w(h(t)) \tag{2.13}$$

for some constant $C > 0$. It is easy to see that inequality (2.13) has a positive bounded solution, which is a contradiction. □

The following two lemmas are concerned with the bounded solutions of second order delay differential inequality (2.6).

Lemma 2.6. *If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} (h(t) - h(s))\, P(s)ds > 1, \tag{2.14}$$

for positive P, then inequality (2.6) *has no positive bounded solutions.*

Proof. Let $w(t)$ be a bounded nonoscillatory solution of inequality (2.6), say $w(t) > 0$ and $w(h(t)) > 0$ for $t \geq t_1 \geq t_0$. Then we obtain

$$w(t) > 0,\ w'(t) < 0 \text{ and } w''(t) \geq 0 \text{ for } t \geq t_1 \geq t_0. \tag{2.15}$$

Now, for $v \geq u \geq t_1$ we have

$$w(u) \geq w(u) - w(v) = -\int_{u}^{v} w'(s)ds \geq (v - u)(-w'(v)). \tag{2.16}$$

For $t \geq s \geq t_1$ setting $u = h(s)$ and $v = h(t)$ in (2.16), we get

$$w(h(s)) \geq (h(t) - h(s))(-w'(h(t))). \tag{2.17}$$

Integrating equation (2.6) from $h(t) \geq t_2$ to t, we have

$$-w'(h(t)) \geq w'(t) - w'(h(t)) \geq \int_{h(t)}^{t} P(s)w(h(s))ds. \tag{2.18}$$

Using (2.17) in (2.18), we have

$$-w'(h(t)) \geq \left(\int_{h(t)}^{t} (h(t) - h(s))P(s)ds\right)(-w'(h(t)))$$

or

$$1 \geq \int_{h(t)}^{t} (h(t) - h(s))P(s)ds. \tag{2.19}$$

We take limsup as $t \to \infty$ of both sides of (2.19), we have a contradiction to condition (2.14) and completes the proof of the lemma. □

Lemma 2.7. *If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \left(\int_{u}^{t} P(s)\right) du > 1, \tag{2.20}$$

then inequality (2.6) *has no positive bounded solutions.*

Proof. Let x be a bounded nonoscillatory solution of inequality (2.6), say $x(t) > 0$ and $x(h(t)) > 0$ for $t \geq t_1 \geq t_0$. As in Lemma 2.6, we obtain (2.15). Integrating (2.6) from u to t

$$w'(t) - w'(u) \geq \int_u^t P(s)w(h(s))ds$$

or

$$-w'(u) \geq \left(\int_u^t P(s)ds\right) w(h(t)).$$

Integrating this inequality from $h(t)$ to t, we get

$$w(h(t)) \geq \left[\int_{h(t)}^t \left(\int_u^t P(s)ds\right) du\right] w(h(t))$$

or

$$1 \geq \left[\int_{h(t)}^t \left(\int_u^t P(s)ds\right) du\right].$$

The rest of the proof is similar to that of Lemma 2.6 and hence is omitted. This completes the proof. □

Theorem 2.8. *Let $\beta \leq 1$ and equation* (1.2) *be nonoscillatory. If condition* (2.1) (*or for every constant $c > 0$, then equation* (2.2) *is oscillatory*) *holds and either condition* (2.14) *or* (2.20) *hold, then equation* (1.1) *is oscillatory.*

Proof. Let x be an eventually positive solution of equation (1.1). Since equation (1.2) is nonoscillatory, then x is of Type (a) or of Type (b) by Lemma 2.1. It follows from Lemma 2.2 or 2.3 that equation (1.1) has no solution of Type (b) and by Lemmas 2.5–2.7 equation (1.1) has no solution of Type (a). This completes the proof. □

Theorem 2.9. *Let $\beta \leq 1$ and equation* (1.2) *be oscillatory. If condition* (2.14) (*or* (2.20)) *holds, then every solution x of equation* (1.1) *is oscillatory or $x''(t)$ is oscillatory.*

Proof. Let x be an eventually positive solution of equation (1.1). Since equation (1.2) is oscillatory, then x is of Type (a) or of Type (c) by Lemma 2.1. By Lemmas 2.5–2.7 equation (1.1) has no solution of Type (a). This completes the proof. □

Example 2.10. Consider the fourth order delay equation

$$x^{(4)}(t) + \frac{1}{4t^2}x^{(2)}(t) + \left(1 - \frac{1}{4t^2}\right) x(t-\pi) = 0. \tag{2.21}$$

Here we let $g(t) = t - \pi$ and $h(t) = t - \frac{\pi}{2}$. All conditions of Theorem 2.8 are satisfied and hence all solutions of equation (2.21) are oscillatory. One such solution is $x(t) = \sin t$. We also note that Theorem 1.1 is applicable to this equation with $g(t) = t$.

Example 2.11. Consider the fourth order delay equation

$$x^{(4)}(t) + 2x^{(2)}(t) + x(t - 2\pi) = 0. \tag{2.22}$$

Here we let $g(t) = t - 2\pi$ and $h(t) = t - \pi$. All conditions of Theorem 2.9 are satisfied and hence all solutions of equation (2.22) are oscillatory. One such solution is $x(t) = \sin t$. We note that Theorem 1.2 is applicable to this equation with $g(t) = t$, i.e.,

$$x^{(4)}(t) + 2x^{(2)}(t) + x(t) = 0,$$

where its solution set is $\{\sin t, \cos t, t \sin t, t \cos t\}$ while

$$x^{(4)}(t) - 2x^{(2)}(t) + x(t) = 0$$

has solution set $\{e^{-t}, e^{t}, te^{-t}, te^{t}\}$. Clearly, the associated second order equation

$$x^{(2)}(t) - 2x(t) = 0$$

is nonoscillatory and Theorem 2.8 fails to apply to this equation because $p(t) = -2 < 0$.

3. GENERAL REMARKS

1. The results of this article are presented in a form which is essentially new and of high degree of generality.
2. It will be of interest to extend the results of this paper to higher order (> 4) equations.
3. It is also of interest to study equation (1.1) with $f(x) = x^{\gamma}$, γ is the ratio of positive odd integers and $1 < \gamma$.

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