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Cauchy Functions for Dynamic Equations on a Measure Chain

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We consider the nth-order linear dynamic equation $Px(t) = \sum_{i=0}^{n} p_i(t)x(\sigma^i(t)) = 0$, where $p_i(t), 0 \leq i \leq n$, are real-valued functions defined on $\mathbb{T}$. We define the Cauchy function $K(t,s)$ for this dynamic equation, and then we prove a variation of constants formula. One of our main concerns is to see how the Cauchy function for an equation is related to the Cauchy functions for the factored parts of the operator P. Finally we consider the equation $Px(t) = \sum_{i=0}^{n} p_i x(\sigma^i(t)) = 0$, where each of the p_i's is a constant, and obtain a formula for the Cauchy function. For our main results we only consider the time scale $\mathbb{T}$ such that every point in $\mathbb{T}$ is isolated.

Key Words: Measure chains; time scales; Cauchy functions.

1. INTRODUCTION

The theory of measure chains was developed by Stefan Hilger and his advisor Bernd Aulbach [6] in 1988 to unify continuous and discrete analysis. In this paper we only consider a special case of measure chains, a so-called *time scale*, which is a nonempty closed subset of the real numbers $\mathbb{R}$. We denote it by $\mathbb{T}$. Later we will define the delta derivative operator $^\Delta$. Choosing the time scale to be the set of real numbers corresponds to the continuous case where $^\Delta$ is the usual derivative, and choosing the time scale to be the set of integers $\mathbb{Z}$ corresponds to the discrete case where $^\Delta$ is the usual forward difference operator Δ defined by

$$\Delta f(t) = f(t+1) - f(t).$$

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There are many other time scales, such as $h\mathbb{Z}$ ($h > 0$), the Cantor set, the set of harmonic numbers $\{\sum_{k=1}^{n} \frac{1}{k}: n \in \mathbb{N}\}$, and so on. One is usually concerned with step-size h, but in some cases one is interested in variable step size. A population of a species where all of the adults die out before the babies are born is an example that could lead to a time scale which is the union of disjoint closed intervals. For a specific example of this type see F. B. Christiansen and T. M. Fenchel [11, p. 7ff]. Any dynamic equation on $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\} := \{q^k: k \in \mathbb{Z}\} \cup \{0\}$ or $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, is called a q-difference equation. These q-difference equations have been studied by Bézivin [7], Trijtzinsky [20], and Zhang [21]. Moreover, Derfel, Romanenko, and Sharkovsky [12] are concerned with the asymptotic behavior of solutions of nonlinear q-difference equations. Bohner and Lutz [9] investigate the asymptotic behavior of dynamic equations on time scales and consider some q-difference equations. Some recent papers concerning dynamic equations on measure chains include Agarwal and Bohner [1, 2], Agarwal et al. [3], Erbe and Hilger [13], Erbe and Peterson [14, 15], Bohner and Eloe [8], Atıcı and Guseinov [5], and Hoffacker [17]. Some preliminary definitions and theorems on measure chains can also be found in the book by Kaymakçalan et al. [18].

A measure chain may or may not be connected, so we define the *forward and backward jump* operators $\sigma, \rho: \mathbb{T} \mapsto \mathbb{T}$ by

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\}, \qquad \rho(t) := \sup\{s < t : s \in \mathbb{T}\},$$

for all $t \in \mathbb{T}$. In this definition we put $\sigma(\emptyset) = \sup \mathbb{T}$ and $\rho(\emptyset) = \inf \mathbb{T}$. If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$ we say t is *left-scattered*. If $\sigma(t) = t$, we say t is *right-dense*, while if $\rho(t) = t$ we say t is *left-dense*. The *graininess* function $\mu: \mathbb{T} \mapsto [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. If $\mathbb{T}$ has a left-scattered maximum m, then we define $\mathbb{T}^\kappa$ to be $\mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. Now we let $f: \mathbb{T} \mapsto \mathbb{R}$ be any function. We define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$\big|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\big| \le \epsilon|\sigma(t) - s|,$$

for all $s \in U$. We call $f^\Delta(t)$ the *delta derivative* of f at t.

Some elementary facts that we will use concerning the delta derivative are contained in the following theorem due to Hilger [16].

THEOREM 1. *Assume $f: \mathbb{T} \mapsto \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we have the following*:

(i) *Every differentiable function is continuous.*

(ii) *If f is continuous at t and $\mu(t) > 0$, then f is differentiable at t with*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) *If f is differentiable at t and $\mu(t) = 0$, then*

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If f is differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Two further examples of such formulas are the product rule, which is given by

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t), \tag{1}$$

where f and g are two differentiable functions, and the quotient rule, which is given by

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}, \tag{2}$$

where f and g are two differentiable functions such that $gg^{\sigma} \neq 0$.

We say $f\colon \mathbb{T} \mapsto \mathbb{R}$ is *right-dense continuous* (rd-continuous) provided f is continuous at each right-dense point $t \in \mathbb{T}$, and whenever $t \in \mathbb{T}$ is left-dense,

$$\lim_{s \to t^-} f(s)$$

exists as a finite number. For example, the function $\mu\colon \mathbb{T} \mapsto \mathbb{R}$ in case $\mathbb{T} = [0, 1] \cup \mathbb{N}$ is rd-continuous but not continuous at 1. Note that if $\mathbb{T} = \mathbb{R}$, then $f\colon \mathbb{R} \mapsto \mathbb{R}$ is rd-continuous on $\mathbb{T}$ if and only if f is continuous on $\mathbb{T}$. Also note that if $\mathbb{T} = \mathbb{Z}$, then any function $f\colon \mathbb{Z} \mapsto \mathbb{R}$ is rd-continuous. A function $F\colon \mathbb{T}^{\kappa} \mapsto \mathbb{R}$ is called a *delta-antiderivative* of $f\colon \mathbb{T} \mapsto \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{\kappa}$. In this case we define the integral of f by

$$\int_a^t f(s)\,\Delta s = F(t) - F(a)$$

for $t \in \mathbb{T}$. Hilger [16] proves that every rd-continuous function on $\mathbb{T}$ has a delta-antiderivative. In the following theorem we give a well-known formula that we use frequently in the later sections.

THEOREM 2. *Assume $f\colon \mathbb{T} \mapsto \mathbb{R}$ is rd-continuous and $t \in \mathbb{T}^{\kappa}$. Then*

$$\int_t^{\sigma(t)} f(\tau)\,\Delta\tau = \mu(t) f(t).$$

We will start with some technical notions given by Hilger [16] to define the exponential function on a general measure chain. He studies the complex exponential function on a measure chain as well. For $h > 0$, let $\mathbb{Z}_h$ be

$$\mathbb{Z}_h := \left\{ z \in \mathbb{C} \colon -\frac{\pi}{h} < \operatorname{Im}(z) \le \frac{\pi}{h} \right\},$$

and let $\mathbb{C}_h$ be defined by

$$\mathbb{C}_h := \left\{ z \in \mathbb{C} \colon z \neq -\frac{1}{h} \right\}.$$

For $h = 0$, let $\mathbb{Z}_0 = \mathbb{C}_0 = \mathbb{C}$, the set of complex numbers. We say that a function $p \colon \mathbb{T} \mapsto \mathbb{R}$ is *regressive* on $\mathbb{T}$ provided

$$1 + \mu(t)p(t) \neq 0 \qquad \text{for all } t \in \mathbb{T}.$$

The set of all regressive functions (Bohner and Peterson [10]) on a time scale $\mathbb{T}$ forms an Abelian group under the addition $\oplus$ defined by

$$p \oplus q := p + q + \mu p q.$$

The additive inverse in this group is denoted by

$$\ominus p := -\frac{p}{1 + \mu p}.$$

We then define subtraction $\ominus$ on the set of regressive functions by

$$p \ominus q := p \oplus (\ominus q).$$

It can be shown that

$$p \ominus q = \frac{p - q}{1 + \mu q}. \tag{3}$$

DEFINITION 3 (Hilger [16]). If $p \colon \mathbb{T} \mapsto \mathbb{R}$ is regressive and rd-continuous, then we define the *exponential function* by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\, \Delta\tau \right)$$

for $t \in \mathbb{T}$, $s \in \mathbb{T}^\kappa$, where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) = \begin{cases} \dfrac{\operatorname{Log}(1 + hz)}{z} & \text{if } h \neq 0 \\ z & \text{if } h = 0. \end{cases}$$

The first-order linear dynamic equation

$$y^\Delta = p(t) y \tag{4}$$

is said to be *regressive* provided p is regressive and rd-continuous on $\mathbb{T}$.

THEOREM 4 (Hilger [16]). *Assume that the dynamic equation* (4) *is regressive and fix* $t_0 \in \mathbb{T}^\kappa$. *Then* $e_p(t, t_0)$ *is the unique solution of the initial value problem*

$$y^\Delta = p(t)y, \qquad y(t_0) = 1, \tag{5}$$

on $\mathbb{T}$.

We next give some important properties of the exponential function which we use frequently in the later sections.

THEOREM 5 (Bohner and Peterson [10]). *Assume* $p, q\colon \mathbb{T} \mapsto \mathbb{R}$ *are rd-continuous and regressive*; *then the following hold*:

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $1/e_p(t, s) = e_p(s, t) = e_{\ominus p}(t, s)$;
(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$ (*semigroup property*);
(v) $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s)$;
(vi) $e_p(t, s)/e_q(t, s) = e_{p\ominus q}(t, s)$.

In Akın et al. [4] the sign of the exponential function $e_p(t, s)$ on a measure chain is determined. A very special case of their result is the following remark.

Remark 6. If $p\colon \mathbb{T}^\kappa \mapsto \mathbb{R}$ is rd-continuous and regressive, then $e_p(t, t_0)$ is real-valued and nonzero on $\mathbb{T}$.

2. CAUCHY FUNCTION AND VARIATION OF CONSTANT FORMULA

In this section we consider

$$Px(t) := \sum_{i=0}^{n} p_i(t)x(\sigma^i(t)) = 0, \tag{6}$$

where $p_i(t)$, $0 \le i \le n$, are real-valued functions defined on $\mathbb{T}$. We will also assume $p_0(t)p_n(t) \neq 0$ on $\mathbb{T}$. In this case we say $Px(t) = 0$ is an *nth-order linear dynamic equation* on a time scale $\mathbb{T}$. We assume throughout the remainder of this paper that every point in our time scale $\mathbb{T}$ is isolated. The following results are motivated by results of Peterson and Schneider [19] for difference equations. It is easy to prove the following theorem.

THEOREM 7. *If* $p_0(t)p_n(t) \neq 0$ *on* $\mathbb{T}$ *and* $t_0 \in \mathbb{T}^{\kappa^{n-1}}$, *then the IVP* (6),

$$x(\sigma^i(t_0)) = x_i, \qquad \text{for } 0 \le i \le n-1,$$

has a unique solution which is defined on all of $\mathbb{T}$.

We now define the important Cauchy function for $Px(t) = 0$.

DEFINITION 8. The *Cauchy function* $K(t, s)$ for (6) is defined on $\mathbb{T} \times \mathbb{T}^{\kappa^{n-1}}$ as follows. For each fixed $s \in \mathbb{T}^{\kappa^{n-1}}$, $K(t, s)$ is the solution of the IVP

$$PK(t, s) = 0, \qquad t \in \mathbb{T}^{\kappa^{n-1}},$$

$$K(\sigma^k(s), s) = 0, \qquad 1 \le k \le n-1, \tag{7}$$

$$K(\sigma^n(s), s) = \frac{1}{p_n(s)\mu(s)}. \tag{8}$$

The Cauchy function is important in the following variation of constant formula.

THEOREM 9 (Variation of Constants Formula). *Assume $f: \mathbb{T}^{\kappa^{n-1}} \mapsto \mathbb{R}$ and $t_0 \in \mathbb{T}^{\kappa^{n-1}}$. Then the solution of the IVP*

$$Px(t) = f(t), \qquad t \in \mathbb{T}^{\kappa^{n-1}},$$
$$x(\sigma^k(t_0)) = 0, \qquad 0 \le k \le n-1,$$

is given by

$$x(t) = \int_{t_0}^{t} K(t, s) f(s)\, \Delta s$$

for $t \in \mathbb{T}$, where $K(t, s)$ is the Cauchy function for (6).

Proof. Let

$$x(t) = \int_{t_0}^{t} K(t, s) f(s)\, \Delta s.$$

First note that

$$x(t_0) = \int_{t_0}^{t_0} K(t_0, s) f(s)\, \Delta s = 0,$$

and for $1 \le k \le n-1$,

$$x(\sigma^k(t_0)) = \int_{t_0}^{\sigma^k(t_0)} K(\sigma^k(t_0), s) f(s)\, \Delta s = 0,$$

where we have used the initial conditions (7). We now show that $Px(t) = f(t)$. To see this, consider

$$\begin{aligned} Px(t) &= \sum_{i=0}^{n} p_i(t) x(\sigma^i(t)) \\ &= \sum_{i=0}^{n} p_i(t) \int_{t_0}^{\sigma^i(t)} K(\sigma^i(t), s) f(s)\, \Delta s \end{aligned}$$

$$= p_0(t)\int_{t_0}^{t} K(t,s)f(s)\,\Delta s + p_1(t)\int_{t_0}^{\sigma(t)} K(\sigma(t),s)f(s)\Delta s$$
$$+ p_2(t)\int_{t_0}^{\sigma^2(t)} K(\sigma^2(t),s)f(s)\,\Delta s$$
$$+ \cdots + p_n(t)\int_{t_0}^{\sigma^n(t)} K(\sigma^n(t),s)f(s)\,\Delta s$$
$$= \sum_{i=0}^{n} p_i(t)\int_{t_0}^{t} K(\sigma^i(t),s)f(s)\,\Delta s + p_1(t)\int_{t}^{\sigma(t)} K(\sigma(t),s)f(s)\,\Delta s$$
$$+ p_2(t)\int_{t}^{\sigma^2(t)} K(\sigma^2(t),s)f(s)\,\Delta s$$
$$+ \cdots + p_n(t)\int_{t}^{\sigma^n(t)} K(\sigma^n(t),s)f(s)\,\Delta s$$
$$= \int_{t_0}^{t} PK(t,s)f(s)\,\Delta s + p_n(t)K(\sigma^n(t),t)f(t)\mu(t)$$
$$= p_n(t)K(\sigma^n(t),t)f(t)\mu(t)$$
$$= f(t),$$

where we have used the initial conditions (7) in the third to the last equation; we have used the fact that $K(t,s)$, for each fixed s, is a solution of $Px(t) = 0$ in the second to the last equation; and we have used (8) in the last equation. ∎

In the next theorem we prove that two iterated integrals are the same.

THEOREM 10. *Assume $a < b \in \mathbb{T}$ and $F(\tau, s)$ is a real-valued function on $\mathbb{T} \times \mathbb{T}$. Then*

$$\int_a^b \int_a^\tau F(\tau,s)\,\Delta s\,\Delta\tau = \int_a^b \int_{\sigma(s)}^b F(\tau,s)\,\Delta\tau\,\Delta s. \tag{9}$$

Proof. Assume $b = \sigma^N(a)$ for some positive integer N. Define

$$G(s) = \int_{\sigma(s)}^{b} F(\tau,s)\,\Delta\tau$$

for $a \le s \le \rho(b)$. Then

$$\int_a^b \int_{\sigma(s)}^b F(\tau,s)\,\Delta\tau\,\Delta s = \int_a^b G(s)\,\Delta s$$
$$= \sum_{i=0}^{N-1} \int_{\sigma^i(a)}^{\sigma^{i+1}(a)} G(s)\,\Delta s$$
$$= \sum_{i=0}^{N-1} G(\sigma^i(a))\mu(\sigma^i(a))$$

$$
\begin{aligned}
&= \sum_{i=0}^{N-1} \mu(\sigma^i(a)) \int_{\sigma^{i+1}(a)}^{b} F(\tau, \sigma^i(a))\, \Delta\tau \\
&= \sum_{i=0}^{N-1} \mu(\sigma^i(a)) \sum_{j=i}^{N-2} \int_{\sigma^{j+1}(a)}^{\sigma^{j+2}(a)} F(\tau, \sigma^i(a))\, \Delta\tau \\
&= \sum_{i=0}^{N-1} \mu(\sigma^i(a)) \sum_{j=i}^{N-2} F(\sigma^{j+1}(a), \sigma^i(a))\mu(\sigma^{j+1}(a)) \\
&= \sum_{i=0}^{N-1} \sum_{j=i}^{N-2} F(\sigma^{j+1}(a), \sigma^i(a))\mu(\sigma^i(a))\mu(\sigma^{j+1}(a)) \\
&= \sum_{i=0}^{N-2} \sum_{j=i}^{N-2} F(\sigma^{j+1}(a), \sigma^i(a))\mu(\sigma^i(a))\mu(\sigma^{j+1}(a)),
\end{aligned}
$$

using our convention on sums in the last equation. Interchanging the order of summation, we get

$$
\int_a^b \int_{\sigma(s)}^b F(\tau, s)\, \Delta\tau\, \Delta s = \sum_{j=0}^{N-2} \sum_{i=0}^{j} F(\sigma^{j+1}(a), \sigma^i(a))\mu(\sigma^i(a))\mu(\sigma^{j+1}(a)).
$$

Changing our index of summation in the outer sum, we get

$$
\begin{aligned}
\int_a^b \int_{\sigma(s)}^b F(\tau, s)\, \Delta\tau\, \Delta s &= \sum_{j=1}^{N-1} \sum_{i=0}^{j-1} F(\sigma^j(a), \sigma^i(a))\mu(\sigma^i(a))\mu(\sigma^j(a)) \\
&= \sum_{j=0}^{N-1} \mu(\sigma^j(a)) \sum_{i=0}^{j-1} \int_{\sigma^i(a)}^{\sigma^{i+1}(a)} F(\sigma^j(a), s)\, \Delta s \\
&= \sum_{j=0}^{N-1} \mu(\sigma^j(a)) \int_a^{\sigma^j(a)} F(\sigma^j(a), s)\, \Delta s \\
&= \sum_{j=0}^{N-1} \int_{\sigma^j(a)}^{\sigma^{j+1}(a)} \int_a^{\tau} F(\tau, s)\, \Delta s\, \Delta\tau \\
&= \int_a^{\sigma^N(a)} \int_a^{\tau} F(\tau, s)\, \Delta s\, \Delta\tau \\
&= \int_a^b \int_a^{\tau} F(\tau, s)\, \Delta s\, \Delta\tau.
\end{aligned}
$$

∎

We next give a formula for the Cauchy function for (6) in the following theorem.

THEOREM 11. *If* $u_1(t), u_2(t), \ldots, u_n(t)$ *are* n *linearly independent solutions of* (6), *then the Cauchy function for* (6) *is given by*

$$K(t,s) = \frac{1}{p_n(s)\mu(s)} \times \frac{\begin{vmatrix} u_1(\sigma(s)) & u_2(\sigma(s)) & \cdots & u_n(\sigma(s)) \\ u_1(\sigma^2(s)) & u_2(\sigma^2(s)) & \cdots & u_n(\sigma^2(s)) \\ \vdots & \vdots & & \vdots \\ u_1(\sigma^{n-1}(s)) & u_2(\sigma^{n-1}(s)) & \cdots & u_n(\sigma^{n-1}(s)) \\ u_1(t) & u_2(t) & \cdots & u_n(t) \end{vmatrix}}{\begin{vmatrix} u_1(\sigma(s)) & u_2(\sigma(s)) & \cdots & u_n(\sigma(s)) \\ u_1(\sigma^2(s)) & u_2(\sigma^2(s)) & \cdots & u_n(\sigma^2(s)) \\ \vdots & \vdots & & \vdots \\ u_1(\sigma^{n-1}(s)) & u_2(\sigma^{n-1}(s)) & \cdots & u_n(\sigma^{n-1}(s)) \\ u_1(\sigma^n(s)) & u_2(\sigma^n(s)) & \cdots & u_n(\sigma^n(s)) \end{vmatrix}}. \tag{10}$$

Proof. Note that $K(\sigma^k(s), s) = 0$ for $1 \le k \le n-1$ and

$$K(\sigma^n(s), s) = \frac{1}{p_n(s)\mu(s)}.$$

For fixed s, we expand the determinant in the numerator in (10) along the last row and get that $K(t,s)$ is a linear combination of solutions of (6) and hence is a solution of (6). ∎

The Wronskian of n functions $y_1, y_2, \ldots, y_n$ defined on $\mathbb{T}$ is defined by

$$W(t) := W(y_1, y_2, \ldots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1(\sigma(t)) & y_2(\sigma(t)) & \cdots & y_n(\sigma(t)) \\ \vdots & \vdots & & \vdots \\ y_1(\sigma^{n-2}(t)) & y_2(\sigma^{n-2}(t)) & \cdots & y_n(\sigma^{n-2}(t)) \\ y_1(\sigma^{n-1}(t)) & y_2(\sigma^{n-1}(t)) & \cdots & y_n(\sigma^{n-1}(t)) \end{vmatrix}$$

for $t \in \mathbb{T}^{\kappa^{n-1}}$.

THEOREM 12 (Abel's Formula for (6)). *Assume $y_1, y_2, \dots, y_n$ are solutions of* (6) *and $t_0 \in \mathbb{T}^\kappa$. Then their Wronskian satisfies*

$$W(t) = W_0 e_{\frac{(-1)^n p_0 - p_n}{p_n \mu}}(t, t_0)$$

for $t \in \mathbb{T}^{\kappa^{n-1}}$, where $W_0 = W(y_1, y_2, \dots, y_n)(t_0)$.

Proof. First note that

$$1 + \mu \frac{(-1)^n p_0 - p_n}{p_n \mu} = \frac{(-1)^n p_0}{p_n} \neq 0.$$

Hence $((-1)^n p_0 - p_n)/p_n \mu$ is regressive, and so $e_{((-1)^n p_0 - p_n)/p_n \mu}(t, t_0)$ is well-defined. Let $y_1, y_2, \dots, y_n$ be solutions of (6). Then

$$W(\sigma(t)) = \begin{vmatrix} y_1(\sigma(t)) & y_2(\sigma(t)) & \cdots & y_n(\sigma(t)) \\ y_1(\sigma^2(t)) & y_2(\sigma^2(t)) & \cdots & y_n(\sigma^2(t)) \\ \vdots & \vdots & & \vdots \\ y_1(\sigma^{n-1}(t)) & y_2(\sigma^{n-1}(t)) & \cdots & y_n(\sigma^{n-1}(t)) \\ y_1(\sigma^n(t)) & y_2(\sigma^n(t)) & \cdots & y_n(\sigma^n(t)) \end{vmatrix}$$

$$= \begin{vmatrix} y_1(\sigma(t)) & y_2(\sigma(t)) & \cdots & y_n(\sigma(t)) \\ y_1(\sigma^2(t)) & y_2(\sigma^2(t)) & \cdots & y_n(\sigma^2(t)) \\ \vdots & \vdots & & \vdots \\ -\frac{p_0(t)}{p_n(t)} y_1(t) & -\frac{p_0(t)}{p_n(t)} y_2(\sigma^n(t)) & \cdots & -\frac{p_0(t)}{p_n(t)} y_n(\sigma^n(t)) \end{vmatrix}$$

$$= \frac{(-1)^n p_0(t)}{p_n(t)} W(t).$$

Since every point in $\mathbb{T}$ is isolated, we get

$$\frac{W(\sigma(t)) - W(t)}{\mu(t)} = \left(\frac{(-1)^n p_0(t) - p_n(t)}{p_n(t)\mu(t)} \right) W(t).$$

Therefore $W = W(y_1, y_2, \dots, y_n)$ is a solution of the IVP

$$W^\Delta = \frac{(-1)^n p_0(t) - p_n(t)}{p_n(t)\mu(t)} W$$

$$W(t_0) = W_0$$

and hence is given by

$$W(t) = W_0 e_{((-1)^n p_0 - p_n)/p_n \mu}(t, t_0).$$

∎

EXAMPLE 13. Consider the second-order dynamic equation

$$x(\sigma^2(t)) - (3 + 2\alpha t)x(\sigma(t)) + 2(1 + \alpha t)x(t) = 0 \tag{11}$$

on $\mathbb{T} := 2^{\mathbb{N}_0} := \{2^m : m \in \mathbb{N}_0\}$, where α is a nonzero, regressive constant. Note that $\sigma(t) = 2t$ and $\mu(t) = t$ on $\mathbb{T} = 2^{\mathbb{N}_0}$. We now find the Cauchy function for (11). It is easy to see that $u_1(t) = 1$ is a solution of (11). We use Abel's formula to find a second linearly independent solution of (11). Let $u_2(t)$ be another solution of (11) satisfying $u_2(1) = \frac{1}{\alpha}$ and $u_2(2) = \frac{1+\alpha}{\alpha}$. Then the Wronskian $W(t)$ of $u_1(t)$ and $u_2(t)$ is given by

$$W(t) = W(1, u_2(t)) = u_2(\sigma(t)) - u_2(t).$$

Note that $W(1) = 1$; hence $u_1(t)$ and $u_2(t)$ are linearly independent solutions of (11). Using Abel's formula for (11), we get

$$\begin{aligned} u_2(\sigma(t)) - u_2(t) &= W(1)e_{((-1)^n p_0 - p_n)/p_n\mu}(t, 1) \\ &= e_{(1+2\alpha t)/t}(t, 1). \end{aligned}$$

Since each point in $\mathbb{T}$ is isolated and $\mu(t) = t$,

$$u_2^{\Delta}(t) = \frac{1}{t}e_{(1+2\alpha t)/t}(t, 1).$$

Then integrating both sides of this equation from 1 to t, we get

$$u_2(t) = \frac{1}{\alpha} + \int_1^t \frac{1}{\tau}e_{(1+2\alpha\tau)/\tau}(\tau, 1)\,\Delta\tau.$$

Notice that $y(t) = t$ solves the IVP

$$y^{\Delta} = \frac{1}{t}y, \qquad y(1) = 1,$$

but we know that $e_{\frac{1}{t}}(t, 1)$ is the unique solution of the above IVP. Therefore $e_{\frac{1}{t}}(t, 1) = t$, and, so,

$$u_2(t) = \frac{1}{\alpha} + \int_1^t e_{\frac{1+2\alpha\tau}{\tau} \ominus \frac{1}{\tau}}(\tau, 1)\,\Delta\tau.$$

Simplifying inside of the integral by using (3), we get

$$u_2(t) = \frac{1}{\alpha} + \int_1^t e_{\alpha}(\tau, 1)\,\Delta\tau = \frac{1}{\alpha}e_{\alpha}(t, 1).$$

Using Theorem 11, the Cauchy function for (11) is given by

$$K(t,s) = \frac{1}{s}\frac{\begin{vmatrix} 1 & \frac{1}{\alpha}e_\alpha(\sigma(s),1) \\ 1 & \frac{1}{\alpha}e_\alpha(t,1) \end{vmatrix}}{\begin{vmatrix} 1 & \frac{1}{\alpha}e_\alpha(\sigma(s),1) \\ 1 & \frac{1}{\alpha}e_\alpha(\sigma^2(s),1) \end{vmatrix}}$$

$$= \frac{1}{s}\frac{e_\alpha(t,1) - e_\alpha(\sigma(s),1)}{e_\alpha(\sigma^2(s),1) - e_\alpha(\sigma(s),1)}$$

$$= \frac{1}{s}\frac{e_\alpha(t,1) - e_\alpha(\sigma(s),1)}{(1+2s\alpha)e_\alpha(\sigma(s),1) - e_\alpha(\sigma(s),1)}.$$

We multiply and divide the right-hand side of the above equation by $e_{\ominus\alpha}(\sigma(s),1)$ to obtain

$$K(t,s) = \frac{1}{s}\frac{e_\alpha(t,1)e_{\ominus\alpha}(\sigma(s),1) - 1}{1+2s\alpha-1}$$

$$= \frac{1}{2s^2\alpha}[e_\alpha(t,1)e_\alpha(1,\sigma(s)) - 1]$$

$$= \frac{1}{2s^2\alpha}[e_\alpha(t,\sigma(s)) - 1].$$

3. CAUCHY FUNCTION FOR FACTORED EQUATIONS

In this section we obtain a formula which gives the Cauchy function for (6) when the operator P can be factored as the composition of two operators.

Assume that we can factor (6) in the form

$$Px(t) = MNx(t), \tag{12}$$

where the operator N is defined by

$$Nx(t) = \sum_{i=0}^{k} q_i(t)x(\sigma^i(t)) \tag{13}$$

and the operator M is defined by

$$Mu(t) = \sum_{i=0}^{n-k} r_i(t)u(\sigma^i(t)) \tag{14}$$

for $1 \le k \le n-1$, with $q_0(t)q_k(t) \neq 0$ and $r_0(t)r_{n-k}(t) \neq 0$ on $\mathbb{T}$.

THEOREM 14. *Assume $Px = 0$ can be written in the factored form* (12), *where the operators N and M are defined by* (13) *and* (14), *respectively. Let $K_N(t,s)$ and $K_M(t,s)$ be the Cauchy functions for $Nx = 0$ and $Mu = 0$, respectively. Then the Cauchy function for* (12) *is given by*

$$K(t,s) = \int_{\sigma(s)}^{t} K_N(t,\tau) K_M(\tau,s)\,\Delta\tau$$

for $t \in \mathbb{T}$, $s \in \mathbb{T}^{\kappa}$.

Proof. Assume $f\colon \mathbb{T}^{\kappa^{n-1}} \mapsto \mathbb{R}$ is rd-continuous. Let $x(t)$ be the solution of the IVP

$$Px(t) = MNx(t) = f(t), \qquad t \in \mathbb{T}^{\kappa^{n-1}},$$
$$x\big(\sigma^k(t_0)\big) = 0, \qquad 0 \le k \le n-1,$$

where $t_0 \in \mathbb{T}^{\kappa^{n-1}}$.

Letting $u(t) = Nx(t)$, we have that $u(t)$ is a solution of the IVP

$$Mu(t) = f(t), \qquad t \in \mathbb{T}^{\kappa^{n-k-1}},$$
$$u\big(\sigma^i(t_0)\big) = 0, \qquad 0 \le i \le n-k-1.$$

Hence by Theorem 9,

$$u(t) = \int_{t_0}^{t} K_M(t,s) f(s)\,\Delta s. \tag{15}$$

Since $x(t)$ is the solution of the IVP

$$Nx(t) = u(t), \qquad t \in \mathbb{T}^{\kappa^{k-1}},$$
$$x(\sigma^i(t_0)) = 0, \qquad 0 \le i \le k-1,$$

using Theorem 9 again, we get

$$x(t) = \int_{t_0}^{t} K_N(t,\tau) u(\tau)\,\Delta\tau.$$

Therefore, using (15) and Theorem 10,

$$\begin{aligned}
x(t) &= \int_{t_0}^{t} K_N(t,\tau) \int_{t_0}^{\tau} K_M(\tau,s) f(s)\,\Delta s\,\Delta\tau \\
&= \int_{t_0}^{t} \int_{t_0}^{\tau} K_N(t,\tau) K_M(\tau,s) f(s)\,\Delta s\,\Delta\tau \\
&= \int_{t_0}^{t} \int_{\sigma(s)}^{t} K_N(t,\tau) K_M(\tau,s) f(s)\,\Delta\tau\,\Delta s \\
&= \int_{t_0}^{t} K(t,s) f(s)\,\Delta s,
\end{aligned}$$

where

$$K(t,s)=\int_{\sigma(s)}^{t} K_N(t,\tau)K_M(\tau,s)\,\Delta\tau.$$

∎

Throughout the rest of this paper, we consider the equation

$$Px(t)=\sum_{i=0}^{n} p_i x(\sigma^i(t))=0, \tag{16}$$

where each of the p_i's is a constant and $p_0 p_n \neq 0$.

We look for a solution of (16) of the form

$$x(t)=e_\lambda(t,t_0),$$

where $\lambda(t)$ is regressive on $\mathbb{T}$ and $t_0 \in \mathbb{T}^{\kappa^{n-1}}$. Letting $x(t)=e_\lambda(t,t_0)$ in (16), we get

$$\left[p_n \prod_{j=0}^{n-1}[1+\mu(\sigma^j(t))\lambda(\sigma^j(t))]+\cdots+p_1[1+\mu(t)\lambda(t)]+p_0\right]e_\lambda(t,t_0)=0.$$

By Remark 6, we know that $e_\lambda(t,t_0)\neq 0$ for all $t\in\mathbb{T}$. Therefore the characteristic equation for (16) is

$$p_n \prod_{j=0}^{n-1}[1+\mu(\sigma^j(t))\lambda(\sigma^j(t))]+\cdots+p_1[1+\mu(t)\lambda(t)]+p_0=0. \tag{17}$$

It follows that if $\lambda(t)$ is regressive on $\mathbb{T}$ and satisfies the characteristic equation (17), then

$$x(t)=e_\lambda(t,t_0)$$

is a solution of (16) on $\mathbb{T}$.

THEOREM 15. *Assume $\lambda_1,\lambda_2,\ldots,\lambda_n$ are distinct, regressive functions on $\mathbb{T}$ satisfying (17) such that if $z_i(t):=1+\mu(t)\lambda_i(t)$ for $1\le i\le n$, then*

$$D(t):=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1(t) & z_2(t) & \cdots & z_n(t) \\ \vdots & \vdots & & \vdots \\ \prod_{j=0}^{n-2} z_1(\sigma^j(t)) & \prod_{j=0}^{n-2} z_2(\sigma^j(t)) & \cdots & \prod_{j=0}^{n-2} z_n(\sigma^j(t)) \end{vmatrix}\neq 0 \tag{18}$$

for $t \in \mathbb{T}^{\kappa^{n-1}}$. *Then the Cauchy function for* (16) *is given by*

$$(19)\quad K(t,s) = \frac{1}{p_n \mu(s)} \times \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1(\sigma(s)) & z_2(\sigma(s)) & \cdots & z_n(\sigma(s)) \\ \vdots & \vdots & & \vdots \\ \prod_{j=1}^{n-2} z_1(\sigma^j(s)) & \prod_{j=1}^{n-2} z_2(\sigma^j(s)) & \cdots & \prod_{j=1}^{n-2} z_n(\sigma^j(s)) \\ e_{\lambda_1}(t,\sigma(s)) & e_{\lambda_2}(t,\sigma(s)) & \cdots & e_{\lambda_n}(t,\sigma(s)) \end{vmatrix}}{D(\sigma(s))}.$$

Proof. Since $\lambda_i(t)$ is a regressive solution of the characteristic equation (17) for $1 \le i \le n$, $x_i(t) := e_{\lambda_i}(t, t_0)$ is a solution of (16) for $1 \le i \le n$. Consider the Wronskian $W(t)$ of these n solutions,

$$\begin{aligned} W(t) &= \begin{vmatrix} e_{\lambda_1}(t,t_0) & e_{\lambda_2}(t,t_0) & \cdots & e_{\lambda_n}(t,t_0) \\ e_{\lambda_1}(\sigma(t),t_0) & e_{\lambda_2}(\sigma(t),t_0) & \cdots & e_{\lambda_n}(\sigma(t),t_0) \\ \vdots & \vdots & & \vdots \\ e_{\lambda_1}(\sigma^{n-1}(t),t_0) & e_{\lambda_2}(\sigma^{n-1}(t),t_0) & \cdots & e_{\lambda_n}(\sigma^{n-1}(t),t_0) \end{vmatrix} \\ &= e_{\lambda_1}(t,t_0)\cdots e_{\lambda_n}(t,t_0) \\ &\quad \times \begin{vmatrix} 1 & \cdots & 1 \\ 1+\mu(t)\lambda_1(t) & \cdots & 1+\mu(t)\lambda_n(t) \\ \vdots & \cdots & \vdots \\ \prod_{j=0}^{n-2}\left[1+\mu^{\sigma^j}(t)\lambda_1^{\sigma^j}(t)\right] & \cdots & \prod_{j=0}^{n-2}\left[1+\mu^{\sigma^j}(t)\lambda_n^{\sigma^j}(t)\right] \end{vmatrix} \\ &= e_{\lambda_1}(t,t_0)\cdots e_{\lambda_n}(t,t_0) \\ &\quad \times \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1(t) & z_2(t) & \cdots & z_n(t) \\ \vdots & \vdots & & \vdots \\ \prod_{j=0}^{n-2} z_1(\sigma^j(t)) & \prod_{j=0}^{n-2} z_2(\sigma^j(t)) & \cdots & \prod_{j=0}^{n-2} z_n(\sigma^j(t)) \end{vmatrix} \\ &= e_{\lambda_1}(t,t_0)\cdots e_{\lambda_n}(t,t_0)D(t) \\ &\ne 0, \end{aligned}$$

using Remark 6 and (18). Then by Theorem 11, we get that

$$K(t,s) = \frac{1}{p_n \mu(s)} \times \frac{\begin{vmatrix} e_{\lambda_1}(\sigma(s),t_0) & e_{\lambda_2}(\sigma(s),t_0) & \cdots & e_{\lambda_n}(\sigma(s),t_0) \\ e_{\lambda_1}(\sigma^2(s),t_0) & e_{\lambda_2}(\sigma^2(s),t_0) & \cdots & e_{\lambda_n}(\sigma^2(s),t_0) \\ \vdots & \vdots & & \vdots \\ e_{\lambda_1}(\sigma^{n-1}(s),t_0) & e_{\lambda_2}(\sigma^{n-1}(s),t_0) & \cdots & e_{\lambda_n}(\sigma^{n-1}(s),t_0) \\ e_{\lambda_1}(t,t_0) & e_{\lambda_2}(t,t_0) & \cdots & e_{\lambda_n}(t,t_0) \end{vmatrix}}{\begin{vmatrix} e_{\lambda_1}(\sigma(s),t_0) & e_{\lambda_2}(\sigma(s),t_0) & \cdots & e_{\lambda_n}(\sigma(s),t_0) \\ e_{\lambda_1}(\sigma^2(s),t_0) & e_{\lambda_2}(\sigma^2(s),t_0) & \cdots & e_{\lambda_n}(\sigma^2(s),t_0) \\ \vdots & \vdots & & \vdots \\ e_{\lambda_1}(\sigma^{n-1}(s),t_0) & e_{\lambda_2}(\sigma^{n-1}(s),t_0) & \cdots & e_{\lambda_n}(\sigma^{n-1}(s),t_0) \\ e_{\lambda_1}(\sigma^n(s),t_0) & e_{\lambda_2}(\sigma^n(s),t_0) & \cdots & e_{\lambda_n}(\sigma^n(s),t_0) \end{vmatrix}}.$$

Since $1+\mu(t)\lambda_i(t) = z_i(t)$ for $1 \le i \le n$ and using Part (ii) of Theorem 5, we obtain that

$$K(t,s) = \frac{1}{p_n \mu(s)} \times \frac{\begin{vmatrix} z_1(s)e_{\lambda_1}(s,t_0) & \cdots & z_n(s)e_{\lambda_n}(s,t_0) \\ \prod_{j=0}^{1} z_1(\sigma^j(s))e_{\lambda_1}(s,t_0) & \cdots & \prod_{j=0}^{1} z_n(\sigma^j(s))e_{\lambda_n}(s,t_0) \\ \vdots & & \vdots \\ \prod_{j=0}^{n-2} z_1(\sigma^j(s))e_{\lambda_1}(s,t_0) & \cdots & \prod_{j=0}^{n-2} z_n(\sigma^j(s))e_{\lambda_n}(s,t_0) \\ e_{\lambda_1}(t,t_0) & \cdots & e_{\lambda_n}(t,t_0) \end{vmatrix}}{\begin{vmatrix} z_1(s)e_{\lambda_1}(s,t_0) & \cdots & z_n(s)e_{\lambda_n}(s,t_0) \\ \prod_{j=0}^{1} z_1(\sigma^j(s))e_{\lambda_1}(s,t_0) & \cdots & \prod_{j=0}^{1} z_n(\sigma^j(s))e_{\lambda_n}(s,t_0) \\ \vdots & & \vdots \\ \prod_{j=0}^{n-1} z_1(\sigma^j(s))e_{\lambda_1}(s,t_0) & \cdots & \prod_{j=0}^{n-1} z_n(\sigma^j(s))e_{\lambda_n}(s,t_0) \end{vmatrix}}.$$

Using Part (iii) and Part (iv) of Theorem 5, we get that

$$K(t,s) = \frac{1}{p_n \mu(s)} \times \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1(\sigma(s)) & z_2(\sigma(s)) & \cdots & z_n(\sigma(s)) \\ \vdots & \vdots & & \vdots \\ \prod_{j=1}^{n-2} z_1(\sigma^j(s)) & \prod_{j=1}^{n-2} z_2(\sigma^j(s)) & \cdots & \prod_{j=1}^{n-2} z_n(\sigma^j(s)) \\ e_{\lambda_1}(t,\sigma(s)) & e_{\lambda_2}(t,\sigma(s)) & \cdots & e_{\lambda_n}(t,\sigma(s)) \end{vmatrix}$$

$$
\Bigg/ \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1(\sigma(s)) & z_2(\sigma(s)) & \cdots & z_n(\sigma(s)) \\ \vdots & \vdots & & \vdots \\ \prod_{j=1}^{n-1} z_1(\sigma^j(s)) & \prod_{j=1}^{n-1} z_2(\sigma^j(s)) & \cdots & \prod_{j=1}^{n-1} z_n(\sigma^j(s)) \end{vmatrix}
$$

$$
= \frac{1}{p_n \mu(s)} \times \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1(\sigma(s)) & z_2(\sigma(s)) & \cdots & z_n(\sigma(s)) \\ \vdots & \vdots & & \vdots \\ \prod_{j=1}^{n-2} z_1(\sigma^j(s)) & \prod_{j=1}^{n-2} z_2(\sigma^j(s)) & \cdots & \prod_{j=1}^{n-2} z_n(\sigma^j(s)) \\ e_{\lambda_1}(t, \sigma(s)) & e_{\lambda_2}(t, \sigma(s)) & \cdots & e_{\lambda_n}(t, \sigma(s)) \end{vmatrix}}{D(\sigma(s))}.
$$

∎

EXAMPLE 16. If $\mathbb{T} = h\mathbb{Z}$, then (16) turns out to be the equation

$$
Px(t) = \sum_{i=0}^{n} p_i x(t + ih) = 0. \tag{20}
$$

We now look for a solution of (20) of the form

$$
x(t) = e_\lambda(t, 0) = (1 + h\lambda)^{\frac{t}{h}},
$$

where λ is a constant and is regressive. Then for $x(t)$ to be a solution we want

$$
p_n(1+h\lambda)^{\frac{t+nh}{h}} + p_{n-1}(1+h\lambda)^{\frac{t+(n-1)h}{h}} + \cdots + p_1(1+h\lambda)^{\frac{t+h}{h}} + p_0(1+h\lambda)^{\frac{t}{h}} = 0.
$$

Dividing by $(1 + h\lambda)^{\frac{t}{h}}$, we get the characteristic equation

$$
p_n(1 + h\lambda)^n + p_{n-1}(1 + h\lambda)^{n-1} + \cdots + p_1(1 + h\lambda) + p_0 = 0. \tag{21}
$$

Letting $z = 1 + h\lambda$ gives us the polynomial equation

$$
p(z) = 0,
$$

where

$$
p(z) = p_n z^n + p_{n-1} z^{n-1} + \cdots + p_1 z + p_0. \tag{22}
$$

It follows that λ_i is a solution of the characteristic equation (21) if and only if $z_i = 1 + h\lambda_i$ is a solution of the polynomial equation (22). Also note that λ_i is regressive if only if $z_i \neq 0$. If z_i is a root of (22), then solving

$$
z_i = 1 + h\lambda_i
$$

for λ_i gives us

$$\lambda_i = \frac{z_i - 1}{h}.$$

It follows that

$$u_i(t) = (1 + h\lambda_i)^{\frac{t}{h}} = z_i^{\frac{t}{h}}$$

is a solution of (20). We claim that if the equation $p(z) = 0$ has n distinct nonzero roots $z_1, z_2, \ldots, z_n$, then the Cauchy function for (20) on $\mathbb{T} = h\mathbb{Z}$ is given by

$$K(t, s) = \frac{1}{h} \sum_{i=1}^{n} \frac{z_i^{\frac{t-s-h}{h}}}{p'(z_i)}.$$

If $\mathbb{T} = h\mathbb{Z}$, then by (19),

$$K(t, s) = \frac{1}{p_n h} \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & & \vdots \\ z_1^{n-2} & z_2^{n-2} & \cdots & z_n^{n-2} \\ z_1^{\frac{t-s-h}{h}} & z_2^{\frac{t-s-h}{h}} & \cdots & z_n^{\frac{t-s-h}{h}} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & & \vdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \end{vmatrix}}.$$

If we expand the determinant in the numerator along the last row and use properties of the Vandermonde determinant, we get the desired result.

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