Existence of Bounded Solutions for Second Order Dynamic Equations

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This Paper is dedicated to Professor Lynn H. Erbe

In this paper, we will give sufficient conditions for a nonhomogeneous dynamic self-adjoint equation on a time scale to have a zero tending solution. We also give sufficient conditions that guarantees that for each constant $C$ there is a unique bounded solution on $[a, \infty)$ with $y(a) = C$.

Keywords: Measure chains; Time scales; Self-adjoint; Dynamic equations

INTRODUCTION

For completeness, we introduce the following concepts related to the notion of time scales. We say $\mathbb{T}$ is a time scale, provided, it is a closed subset of the

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real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. We also assume throughout this paper that $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$.

**Definition 1** We define the forward jump operator $\sigma$, for $t \in \mathbb{T}$, by

$$\sigma(t) = \inf \{ \tau > t : \tau \in \mathbb{T} \},$$

and the backward jump operator $\rho$, for $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, by

$$\rho(t) = \sup \{ \tau < t : \tau \in \mathbb{T} \}.$$

If $\sigma(t) > t$, we say $t$ is right-scattered, while if $\rho(t) < t$ we say $t$ is left-scattered. If $\sigma(t) = t$ we say $t$ is right-dense, while if $\rho(t) = t$ we say $t$ is left-dense. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be right-dense continuous provided $f$ is continuous at right-dense points in $\mathbb{T}$ and at left-dense points in $\mathbb{T}$, left hand limits exist and are finite. We shall also use the notation $\mu(t) : \sigma(t) - t$ and we call $\mu$ the graininess function. Finally, if $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function, then we define the function $f^\omega : \mathbb{T} \rightarrow \mathbb{R}$ by

$$f^\omega(t) = f(\sigma(t)) \text{ for all } t \in \mathbb{T}.$$ 

i.e. $f^\omega = f^\sigma$. Similarly, $f^\rho = f^\rho \rho$.

**Definition 2** We define the interval in $\mathbb{T}$

$$[a, \infty) := \{ t \in \mathbb{T} \text{ such that } t \geq a \}.$$ 

The notion of a measure chain was introduced by Hilger [10]. Related work on the calculus of measure chains may be found in Refs. [2,3,7-9]. For an introduction to dynamic equations on time scales see Refs. [1,5,6,11].

**Definition 3** Assume $x : \mathbb{T} \rightarrow \mathbb{R}$ and fix $t \in \mathbb{T}$, then we define $x^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ such that

$$\| x(\sigma(t)) - x(s) - x^\Delta(t)[\sigma(t) - s] \| \leq \varepsilon |\sigma(t) - s|,$$

for all $s \in U$. We call $x^\Delta(t)$ the delta derivative of $x(t)$ at $t$. 
It can be shown that if $x: \mathbb{T} \rightarrow \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and $t$ is right-scattered, then
\[ x^\Delta(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}. \]

Note that if $\mathbb{T} = \mathbb{Z}$, the integers, then
\[ x^\Delta(t) = \Delta x(t) = x(t + 1) - x(t). \]

If $t$ is right-dense, then
\[ x^\Delta(t) = \lim_{s \to t} \frac{x(t) - x(s)}{t - s}. \]

if the limit exists. In particular, if $\mathbb{T} = \mathbb{R}$, the real numbers, then $x^\Delta(t) = x'(t)$.

Also an integral $\int_a^b h(t)\Delta t$ can be defined (see Ref. [6]). It turns out if $\mathbb{T} = \mathbb{R}$, then
\[ \int_a^b h(t)\Delta t = \int_a^b h(t)dt \]
is the Riemann integral and if $\mathbb{T} = \mathbb{Z}$ and $a < b$ are integers, then
\[ \int_a^b h(t)\Delta t = \sum_{t=a}^{b-1} h(t). \]

MAIN RESULTS

In this section, we will state and prove out main results. We will mainly be concerned with the linear nonhomogeneous dynamic equation with variable coefficients in self-adjoint form:
\[ (p(t)y^\Delta(t))^\Delta + q(t)y^\sigma(t) = f(t) \quad (1) \]
where $p$, $q$, and $f$ are rd-continuous functions on $\mathbb{T}$ and $p$ is a positive function.

The following result guarantees the existence of a solution of Eq. (1) converging to zero as $t \to \infty$ independent of whether it is oscillatory or nonoscillatory.
THEOREM 4 If

(i) \( p(t) > 0, \quad q(t) \geq 0 \) for all \( t \in [a, \infty) \),

(ii) \( \int_a^\infty \frac{1}{p(\tau)} \Delta \tau < \infty \),

(iii) \( \int_a^\infty q(\tau) \sigma^\sigma(\tau) \Delta \tau < \infty \), where \( P(t) = \int_t^\infty \frac{1}{p(s)} \Delta s \),

(iv) \( \int_a^\infty f(\tau) \Delta \tau < \infty \),

then Eq. (1) has a solution which converges to zero as \( t \to \infty \).

Proof Let

\[
F(t) = \int_t^\infty f(\tau) \Delta \tau
\]

and

\[
K(t) = \int_t^\infty \frac{F(s)}{p(s)} \Delta s.
\]  \hspace{1cm} (2)

where \( K \) is well defined follows from (ii) and (iv). Therefore \( K(t) \to 0 \) as \( t \to \infty \). Also, \( P(t) \to 0 \) as \( t \to \infty \) and \( \int_t^\infty q(\tau) \sigma^\sigma(\tau) \Delta \tau \to 0 \) as \( t \to \infty \) because of (ii) and (iii), respectively.

By (iii) choose \( T \in [a, \infty) \) sufficiently large so that

\[
\alpha := \int_T^\infty q(\tau) \sigma^\sigma(\tau) \Delta \tau \in (0, 1).
\]  \hspace{1cm} (3)

Let \( X \) be the Banach Space of all continuous functions \( x : [T, \infty) \to \mathbb{R} \) which converge to zero with the norm \( \| \cdot \| \) defined by

\[
\|y\| = \sup \{ |y(t)| : t \in [T, \infty) \},
\]

and define the operator \( A \) on \( X \) by

\[
A y(t) = K(t) + L y(t), \quad y \in X,
\]  \hspace{1cm} (4)

where \( K \) is defined by Eq. (2) and \( L \) is the operator defined by

\[
L y(t) = P(t) \int_T^\infty q(\tau) y^\sigma(\tau) \Delta \tau + \int_t^\infty q(\tau) \sigma^\sigma(\tau) y^\sigma(\tau) \Delta \tau
\]  \hspace{1cm} (5)
for all $t \in [T, \infty)$. It is clear that $Ay$ is continuous on $[T, \infty)$. Hence to show $A : X \rightarrow X$ it remains to show that $\lim_{t \to \infty} Ay(t) = 0$. Since $K(t) \rightarrow 0$ as $t \to \infty$, we need only to show that $Ly(t) \rightarrow 0$ as $t \to \infty$. To show this, it suffices to show that if

$$x(t) = P(t) \int_T^t q(\tau)y^\sigma(\tau)\Delta \tau,$$

then $\lim_{t \to \infty} x(t) = 0$. To see this, let $\varepsilon > 0$ be given. Choose $t_0 \in T$ such that $t_0 \geq T$ and

$$|y^\sigma(t)| < \varepsilon \quad \text{for all } t > t_0. \quad (6)$$

For this $t_0$, set

$$\beta = \left| \int_{t_0}^0 q(\tau)y^\sigma(\tau)\Delta \tau \right| \quad (7)$$

and since $P(t) \rightarrow 0$ as $t \to \infty$ we can choose $t_1 \in [t_0, \infty)$ such that

$$P(t_1)\beta < \varepsilon \quad (8)$$

for all $t \geq t_1$. Then for all $t \geq t_1,$

$$|x(t)| = \left| P(t) \int_T^t q(\tau)y^\sigma(\tau)\Delta \tau \right| = \left| P(t) \int_{t_0}^0 q(\tau)y^\sigma(\tau)\Delta \tau + P(t) \int_{t_0}^t q(\tau)y^\sigma(\tau)\Delta \tau \right|$$

$$\leq P(t) \int_{t_0}^0 q(\tau)y^\sigma(\tau)\Delta \tau + P(t) \int_{t_0}^t q(\tau)|y^\sigma(\tau)|\Delta \tau$$

$$= P(t)\beta + \int_{t_0}^t q(\tau)P^\sigma(\tau)|y^\sigma(\tau)|\Delta \tau < \varepsilon + \varepsilon \int_{t_0}^t q(\tau)P^\sigma(\tau)\Delta \tau$$

$$< \varepsilon(1 + \alpha)$$

since $P$ is decreasing and we used Eqs. (7), (8), (6), and (3), respectively, (here we used $P(t) \int_{t_0}^t q(\tau)\Delta \tau < \int_{t_0}^t q(\tau)P^\sigma(\tau)\Delta \tau$ but this is easy to verify).

Hence $\lim_{t \to \infty} x(t) = 0$ and so $A : X \rightarrow X$. Next we show that $A$ is a contraction.
mapping on $X$. Let $y, z \in X$, $t \geq T$ and consider

$$|Ay(t) - Az(t)| = |L_y(t) - L_z(t)| \leq P(t) \int_T^t q(\tau) |y^\sigma(\tau) - z^\sigma(\tau)| \Delta \tau$$

$$+ \int_T^t q(\tau) P^\sigma(\tau)|y^\sigma(\tau) - z^\sigma(\tau)| \Delta \tau$$

$$\leq \int_T^t q(\tau) P^\sigma(\tau)|y^\sigma(\tau) - z^\sigma(\tau)| \Delta \tau + \int_T^t q(\tau) P^\sigma(\tau)|y^\sigma(\tau) - z^\sigma(\tau)| \Delta \tau$$

$$= \int_T^t q(\tau) P^\sigma(\tau)|y^\sigma(\tau) - z^\sigma(\tau)| \Delta \tau \leq \|y - z\| \int_T^t q(\tau) P^\sigma(\tau) \Delta \tau = \alpha \|y - z\|$$

since $P$ is decreasing and by Eq. (3). Therefore

$$\|Ay - Az\| \leq \alpha \|y - z\|$$

for all $y, z \in X$ and so $A$ is a contraction mapping on $X$. Hence by the Banach Fixed Point Theorem, $A$ has a unique fixed point $y \in X$. Since $y = Ay$, we have $y(t)$ converges to $0$ as $t \to \infty$. It remains to show that Eq. (9) is a solution of Eq. (1). Since $y = Ay$,

$$y(t) = K(t) + P(t) \int_T^t q(\tau) y^\sigma(\tau) \Delta \tau + \int_T^t q(\tau) P^\sigma(\tau) y^\sigma(\tau) \Delta \tau \quad (9)$$

for all $t \geq T$. Taking the derivative of both sides we get

$$y^\Delta(t) = K^\Delta(t) + P^\sigma(t)q(t)y^\sigma(t) + P^\Delta(t) \int_T^t q(\tau) y^\sigma(\tau) \Delta \tau + q(t)P^\sigma(t)y^\sigma(t)$$

$$y^\Delta(t) = -\frac{F(t)}{p(t)} \frac{1}{p(t)} \int_T^t q(\tau) y^\sigma(\tau) \Delta \tau$$

and so

$$p(t)y^\Delta(t) = -F(t) - \int_T^t q(\tau) y^\sigma(\tau) \Delta \tau$$

and therefore

$$(p(t)y^\Delta(t))^\Delta = f(t) - q(t)y^\sigma(t).$$

So we get the desired result.
DEFINITION 5 A function $x$ is called eventually positive (eventually negative) provided there is a $T \in [a, \infty)$ such that $x(t) \geq 0$ ($x(t) \leq 0$) for all $t \in [T, \infty)$.

COROLLARY 6 Assume that (i)-(iv) of Theorem 4 hold. If $f$ is eventually positive (eventually negative), then Eq. (1) has an eventually positive (eventually negative) solution converging to zero as $t \to \infty$.

Proof Assume $f$ is eventually positive. Without loss of generality we can assume that $f(t) \geq 0$ for all $t \in [T, \infty)$, where $T$ is as in the proof of Theorem 4. Let $A$ be defined by Eqs. (4) and (5) and consider the closed subset $S^+ \subseteq X$ defined by

$$S^+ = \{ y \in X : y(t) \geq 0 \text{ for all } t \in [T, \infty) \}. \tag{10}$$

As $K(t) \geq 0$ for all $t \in [T, \infty)$, $AS^+ \subseteq S^+$. By the same argument as in the proof of Theorem 1, one can show that $A$ is a contraction on $S^+$. Therefore we get the desired result in the eventual positive case. The eventual negative case is similar.

REMARK 7 Since $(1/t)^A = -(1/t\sigma(t))$, $\int_{1}^{\infty} (1/t\sigma(t))dt = 1$.

Example 8 Assume $\sup T = \infty$. Consider the dynamic equation

$$(\sigma(t)x^A(t))^A + \frac{1}{t\sigma(t)}x^\sigma(t) - \frac{1}{t(\sigma(t))^2}$$

on $[1, \infty) \cap T$. Using Remark 7 we see that the hypotheses of Theorem 1 are satisfied. One of its solutions (use Remark 7) on $T$ is $x(t) = 1/t$ which converges to zero as $t \to \infty$.

We now consider the analogue of Theorem 4 for the self-adjoint equation

$$[p(t)y^A(t)]^\sigma + q(t)y(t) = f(t) \tag{11}$$

which was introduced by Atici and Guseinov [5]. When considering this equation we assume that $f : \mathbb{T} \to \mathbb{R}$ is continuous, $p : \mathbb{T} \to \mathbb{R}$ is continuous and positive, and $q : \mathbb{T} \to \mathbb{R}$ is continuous. For the definition of the nabla derivative $\nabla$ and the nabla integral $\int_{a}^{b} h(t)\nabla t$ see Refs. [4,6].
THEOREM 9  If

(i) \( p(t) > 0, q(t) \geq 0 \) for all \( t \in [a, \infty) \),

(ii) \( \int_a^\infty \frac{1}{p(\tau)} \Delta \tau < \infty, \quad \int_a^\infty \frac{1}{p(\tau)} \Delta \tau < \infty \),

(iii) \( \int_a^\infty q^\sigma(\tau) P^\sigma(\tau) \Delta \tau < \infty \), where \( P(t) := \int_t^\infty \frac{1}{p(s)} \Delta s \),

(iv) \( \int_a^\infty f(\tau) \Delta \tau < \infty \),

then Eq. (1) has a solution which converges to zero as \( t \to \infty \).

Proof  The proof is very similar to the proof of Theorem 4 so we will just indicate how we define some things differently for this case and do the last part of the proof. In this proof \( F, P \), and \( K \) are defined by

\[ F(t) := \int_t^\infty f(s) \Delta s, \quad P(t) := \int_t^\infty \frac{1}{p(s)} \Delta s, \quad K(t) := \int_t^\infty \frac{F(s)}{p(s)} \Delta s \]

and \( L \) and \( A \) are defined by

\[ Ly(t) := P(t) \int_t^\infty q(\tau) y(\tau) \Delta \tau + \int_t^\infty P^\sigma(\tau) q^\sigma(\tau) y^\sigma(\tau) \Delta \tau \]

and

\[ Ay(t) = Ly(t) + K(t), \]

respectively, where \( T \) is sufficiently large. We just show that if \( y \) is a fixed point of \( A \), then \( y \) is a solution of Eq. (11). To see this consider

\[ y(t) = Ay(t) = P(t) \int_t^\infty q(\tau) y(\tau) \Delta \tau + \int_t^\infty P^\sigma(\tau) q^\sigma(\tau) y^\sigma(\tau) \Delta \tau + K(t). \]

Taking the delta derivative of both sides and using the formula (see Ref. [4] or [6])

\[ \left( \int_a^\tau h(s) \Delta s \right)^\Delta = h(\sigma(t)) \]
we get
\[ y^\Delta(t) = P_\sigma(t)q_\sigma(t)y_\sigma(t) - \frac{1}{p(t)} \int_t^\infty q(\tau)y(\tau)\nabla\tau - P_\sigma(t)q_\sigma(t)y_\sigma(t) \]
\[ - \frac{1}{p(t)} \int_t^\infty f(\tau)\nabla\tau. \]

It follows that
\[ p(t)y^\Delta(t) = -\int_t^\infty q(\tau)y(\tau)\nabla\tau - \int_t^\infty f(\tau)\nabla\tau. \]

Taking the nabla derivative of both sides we get the desired result
\[ [p(t)y^\Delta(t)]^\nabla = -q(t)y(t) + f(t). \]

In the next result we relax the positivity of \( q \) and the condition on \( f \) guaranteeing the existence of bounded solutions for Eq. (1).

**THEOREM 10** Assume there are constants \( \mu, \bar{\mu} \) such that \( 0 < \mu \leq \mu(t) \leq \bar{\mu} \) for all \( t \in [a, \infty) \).

If the conditions
(i) \( 0 < \alpha \leq p(t) \leq \beta, \) for all \( t \in [a, \infty) \),

(ii) either
\[ q(t) \leq \gamma < 0, \] for all \( t \in [a, \infty) \),

or there is a constant \( \delta > 2 \) such that
\[ \frac{q(t)\mu^2(t)\mu^\sigma(t)}{p^{\sigma}(t)\mu(t) + p(t)\mu^\sigma(t)} \geq \delta > 2 \] for all \( t \in [a, \infty) \),

(iii) \( f \) is bounded on \( [a, \infty) \), hold, then for each number \( C \) there is a unique bounded solution of Eq. (1) with \( y(a) = C \).

**Proof** Equation (1) can be written in the form:
\[ p^{\sigma}(t)\mu(t)y^{\sigma^2}(t) - [p^{\sigma}(t)\mu(t) + p(t)\mu^\sigma(t)]y^{\sigma}(t) - q(t)\mu^2(t)\mu^\sigma(t)y^\sigma(t) \]
\[ + p(t)\mu^\sigma(t)y(t) = \mu^2(t)\mu^\sigma(t)f(t) \]
hence
\[ y^\sigma(t) = \frac{p^{\sigma}(t)\mu(t)y^{\sigma^2}(t) + p(t)\mu^\sigma(t)y(t) - \mu^2(t)\mu^\sigma(t)f(t)}{p^{\sigma}(t)\mu(t) + p(t)\mu^\sigma(t) - q(t)\mu^2(t)\mu^\sigma(t)}. \]
Let \( t = \rho(s) \), then

\[
\gamma(s) = \frac{p(s)\mu^p(s)y^\rho(s) + p^\rho(s)\mu(s)y^\rho(s) - \mu^2(\rho(s))\mu(s)f^\rho(s)}{p(s)\mu^p(s) + p^\rho(s)\mu(s) - q^\rho(s)\mu^2(\rho(s))\mu(s)},
\]

for \( s \in [\sigma(a), \infty) \).

Consider the Banach Space \( X \) of bounded functions on \([a, \infty)\) with the norm \( ||y|| = \sup |y(t)| \ t \in [a, \infty) \). Define the operator \( T \) by

\[
Ty(a) = C,
\]

\[
Ty(s) = \frac{p(s)\mu^p(s)y^\rho(s) + p^\rho(s)\mu(s)y^\rho(s) - \mu^2(\rho(s))\mu(s)f^\rho(s)}{p(s)\mu^p(s) + p^\rho(s)\mu(s) - q^\rho(s)\mu^2(\rho(s))\mu(s)},
\]

for all \( s \geq \sigma(a) \).

It is clear that \( T : X \mapsto X \). Since the numerator is bounded above and the denominator is bounded away from zero, \( T \) is bounded.

Consider the case when condition (12) is satisfied. In this case \( p(s)\mu^p(s) + p^\rho(s)\mu(s) - q^\rho(s)\mu^2(\rho(s))\mu(s) > 0 \). We now show that \( T \) is a contraction mapping on \( X \). Let \( y, z \in X \), and \( s \geq \sigma(a) \). Then

\[
|Ty(s) - Tz(s)| = \left| \frac{p(s)\mu^p(s)y^\rho(s) - z^\rho(s) + p^\rho(s)\mu(s)(y^\rho(s) - z^\rho(s))}{p(s)\mu^p(s) + p^\rho(s)\mu(s) - q^\rho(s)\mu^2(\rho(s))\mu(s)} \right|
\]

\[
\leq \frac{p(s)\mu^p(s)(y^\rho(s) - z^\rho(s))}{p(s)\mu^p(s) + p^\rho(s)\mu(s) - q^\rho(s)\mu^2(\rho(s))\mu(s)}
\]

\[
\leq \frac{p(s)\mu^p(s) + p^\rho(s)\mu(s)}{p(s)\mu^p(s) + p^\rho(s)\mu(s) - q^\rho(s)\mu^2(\rho(s))\mu(s)} \ ||y - z||
\]

for \( s \in [\sigma(a), \infty) \). Consider

\[
\frac{p(s)\mu^p(s) + p^\rho(s)\mu(s)}{p(s)\mu^p(s) + p^\rho(s)\mu(s) - q^\rho(s)\mu^2(\rho(s))\mu(s)} = \frac{1}{1 - \left( \frac{q^\rho(s)\mu^2(\rho(s))\mu(s)}{p(s)\mu^p(s) + p^\rho(s)\mu(s)} \right)}.
\]

But from (i), Eq. (12) and the assumptions on \( \mu(t) \), we have

\[
\frac{q^\rho(s)\mu^2(\rho(s))\mu(s)}{p(s)\mu^p(s) + p^\rho(s)\mu(s)} \leq \frac{\gamma\mu^3}{2\beta\mu} = \kappa < 0
\]
for $s \in [\sigma(a), \infty)$. Therefore

$$|Ty(s) - Tz(s)| \leq \frac{p(s)\mu^p(s) + p^p(s)\mu(s)}{p(s)\mu^p(s) + p^p(s)\mu(s) - q^p(s)\mu^2(p(s))\mu(s)} \leq \frac{1}{1 - \kappa} < 1$$

for all $s \in [Ty(a), \infty)$. If $s = a$, then

$$|Ty(a) - Tz(a)| = 0 \leq \frac{1}{1 - \kappa} \|y - z\|.$$

Therefore

$$|Ty(s) - Tz(s)| < \frac{1}{1 - \kappa} \|y - z\|, \text{ for all } s \in [a, \infty).$$

This implies that

$$\|Ty - Tz\| \leq \frac{1}{1 - \kappa} \|y - z\|$$

and so $T$ is a contraction mapping on $X$. Hence by the Banach Fixed Point Theorem, $T$ has a unique fixed point $y$. It follows that

$$y(a) = Ty(a) = C,$$

$$y(s) = Ty(s) = \frac{p(s)\mu^p(s)y^\sigma(s) + p^p(s)\mu(s)y^p(s) - \mu^2(p(s))\mu(s)f^p(s)}{p(s)\mu^p(s) + p^p(s)\mu(s) - q^p(s)\mu^2(p(s))\mu(s)},$$

for all $s \geq \sigma(a)$

is the unique bounded solution of Eq. (1) satisfying $y(a) = C$.

Finally consider the case when Eq. (13) is satisfied. Again we will show that $T$ is a contraction mapping on $X$. Let $y, z \in X$, and $s \geq \sigma(a)$. Then

$$|Ty(s) - Tz(s)| = \frac{|p(s)\mu^p(s)(y^\sigma(s) - z^\sigma(s)) + p^p(s)\mu(s)(y^p(s) - z^p(s))|}{p(s)\mu^p(s) + p^p(s)\mu(s) - q^p(s)\mu^2(p(s))\mu(s)}$$

$$\leq \frac{|p(s)\mu^p(s)(y^\sigma(s) - z^\sigma(s)) + p^p(s)\mu(s)(y^p(s) - z^p(s))|}{|p(s)\mu^p(s) + p^p(s)\mu(s) - q^p(s)\mu^2(p(s))\mu(s)|}$$

$$\leq \frac{p(s)\mu^p(s) + p^p(s)\mu(s)}{|p(s)\mu^p(s) + p^p(s)\mu(s) - q^p(s)\mu^2(p(s))\mu(s)|} \|y - z\|.$$
on \([\sigma(a), \infty)\). Consider

\[
\frac{p(s)\mu^p(s) + p^p(s)\mu(s)}{p(s)\mu^p(s)[p^p(s)\mu(s) - q^p(s)\mu^2(p(s))\mu(s)]} = \frac{1}{1 - \left( \frac{q^p(s)\mu^2(p(s))\mu(s)}{p(s)\mu^p(s) + p^p(s)\mu(s)} \right)}.
\]

But from (ii), Eq. (13) we have

\[
2 < \delta \leq \frac{q^p(s)\mu^2(p(s))\mu(s)}{p(s)\mu^p(s) + p^p(s)\mu(s)}
\]

for \(s \in [\sigma(a), \infty)\). This implies that

\[
\frac{1}{1 - \delta} \leq \frac{1}{1 - \left( \frac{q^p(s)\mu^2(p(s))\mu(s)}{p(s)\mu^p(s) + p^p(s)\mu(s)} \right)} < 0
\]

for \(s \in [\sigma(a), \infty)\). Therefore

\[
|Ty(s) - Tz(s)| \leq \left| \frac{p(s)\mu^p(s)[p^p(s)\mu(s)]}{p(s)\mu^p(s)[p^p(s)\mu(s) - q^p(s)\mu^2(p(s))\mu(s)]} \right| \leq \frac{1}{\delta - 1} < 1
\]

for all \(s \in [\sigma(a), \infty)\). If \(s = a\), then

\[
|Ty(a) - Tz(a)| = 0 \leq \frac{1}{\delta - 1} \|y - z\|
\]

Therefore

\[
|Ty(s) - Tz(s)| \leq \frac{1}{\delta - 1} \|y - z\|, \quad \text{for all } s \in [a, \infty).
\]

Hence

\[
\|Ty - Tz\| \leq \frac{1}{\delta - 1} \|y - z\|
\]

and so \(T\) is a contraction mapping on \(X\). By the Banach Fixed Point Theorem, \(T\) has a unique fixed point \(y\). It follows that

\[
y(a) = Ty(a) = C,
\]
SECOND ORDER DYNAMIC EQUATIONS

\[ y(s) = Ty(s) = \frac{p(s)\mu(s)y''(s) + p''(s)\mu(s)y'(s) - \mu^2(s)z(s)f(s)}{p(s)\mu(s) + p''(s)\mu(s) - q(s)\mu^2(s)z(s)} \]

for all \( s \geq \sigma(a) \)

is the unique bounded solution of Eq. (1) satisfying \( y(a) = C \).

\[ \square \]

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References