Halanay type inequalities on time scales with applications

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ABSTRACT

This paper aims to introduce Halanay type inequalities on time scales. By means of these inequalities we derive new global stability conditions for nonlinear dynamic equations on time scales. Giving several examples we show that besides generalization and extension to q-difference case, our results also provide improvements for the existing theory regarding differential and difference inequalities, which are the most important particular cases of dynamic inequalities on time scales.

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1. Introduction and preliminaries

Stability analysis of dynamical systems using differential and difference inequalities attracted a prominent attention in the existing literature (see [1–12] and references therein). For stability analysis of the delay differential equation

$$x'(t) = -px(t) + qx(t - \tau), \quad \tau > 0,$$

Halanay proved the following result.

Lemma 1 (*Halanay [2]*)**.** *If*

$$f'(t) \leq -\alpha f(t) + \beta \sup_{s \in [t-\tau, t]} f(s) \quad \text{for } t \geq t_0$$

and $\alpha > \beta > 0$, *then there exist* $\gamma > 0$ *and* $K > 0$ *such that*

$$f(t) \leq K e^{-\gamma(t-t_0)} \quad \text{for } t \geq t_0.$$

In 2000, Mohamad and Gopalsamy gave the next theorem:

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Theorem 1 (*[12]*)**.** *Let x be a nonnegative function satisfying*

$$x'(t) \leq -a(t)x(t) + b(t) \left(\sup_{s \in [t-\tau(t),t]} x(s) \right), \quad t \geq t_0$$

$$x(s) = |\varphi(s)| \quad for\ s \in [t_0 - \tau^*, t_0],$$

where $\tau(t)$ denotes a nonnegative continuous and bounded function defined for $t \in \mathbb{R}$ and $\tau^* = \sup_{t \in \mathbb{R}} \tau(t)$; $\varphi(s)$ is continuous and defined on $[t_0 - \tau^*, t_0]$; $a(t)$ and $b(t)$, $t \in \mathbb{R}$, denote nonnegative, continuous and bounded functions. Suppose

$$a(t) - b(t) \geq L, \quad t \in \mathbb{R},$$

where $L = \inf_{t \in \mathbb{R}} (a(t) - b(t)) > 0$. Then there exists a positive number λ such that

$$x(t) \leq \left(\sup_{s \in [t_0 - \tau^*, t_0]} x(s) \right) e^{-\lambda(t-t_0)} \quad for\ t > t_0.$$

Afterwards, numerous variants of Halanay's inequality have been treated in the literature. Stability analysis of differential equations using Halanay type inequalities has been studied in [2,4,5]. For stability analysis of difference equations using Halanay inequality one may consult with [6–9]. A comprehensive review on the recent developments in discrete and continuous Halanay type inequalities can be found in [10,11]. A time scale is an arbitrary nonempty closed subset of reals. Stability analysis of dynamics equations on time scales using Lyapunov functionals has been studied in [13–20]. To the best of our knowledge, Halanay type inequalities on time scales and stability analysis using them have not been investigated elsewhere before this study. One of the aims of this paper is to fill this gap and show how Halanay inequalities on time scales can be used for the stability analysis of dynamic equations.

In this paper, we employ the shift operators $\delta_\pm$ to construct delay dynamic inequalities on time scales. Using these dynamic inequalities we derive Halanay type inequalities for dynamic equations on time scales. By means of Halanay inequalities and the properties of exponential function on time scales (see Lemma 3) we propose new conditions that lead to stability for nonlinear dynamic equations on time scales. Main contribution of this paper can be outlined as follows:

- Construction of Halanay type inequalities on time scales,
- Investigation of global stability of delay dynamic equations on time scales using Halanay inequality,
- Improvement of the existing results for differential and difference equations which are the most important particular cases of our problem (we highlight this improvement by Remarks 2–4).

In [21], Halanay inequalities are used to derive sufficient conditions for the existence of periodic solutions of delayed cellular neural networks with impulsive effects. Motivated by the study [21], we note that the results obtained in this paper can also be employed in another research regarding the derivation of sufficient conditions for the existence of (uniformly asymptotically stable) periodic solutions of some nonlinear scalar systems on time scales.

We organize the paper as follows: First and second sections are devoted to preliminary results of theory of time scales and shift operators on time scales, respectively. In the third section, we use the shift operators on time scales to construct delay functions and a general form of delay dynamic equations, and obtain some dynamic inequalities. We finalize our study by providing sufficient conditions for stability of nonlinear dynamic equations on time scales.

Hereafter, we give some basic results that will be used in our further analysis.

To indicate a time scale (a nonempty closed subset of reals) we use the notation $\mathbb{T}$. We classify the points of a time scale $\mathbb{T}$ by using the forward jump and backward jump operators defined by

$$\sigma(t) := \inf \{s \in \mathbb{T} : s > t\} \tag{1.1}$$

and

$$\rho(t) := \sup \{s \in \mathbb{T} : s < t\},$$

respectively. A point t in $\mathbb{T}$ is said to be right-scattered (right-dense) if $\sigma(t) > t$ ($\sigma(t) = t$). We say $t \in \mathbb{T}$ is left-scattered (left-dense) if $\rho(t) < t$ ($\rho(t) = t$). If $\rho(t) < t < \sigma(t)$, then $t \in \mathbb{T}$ is called isolated point. The set $\mathbb{T}^\kappa$ is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^\kappa = \mathbb{T}$. The delta derivative of a function $f : \mathbb{T} \to \mathbb{R}$, defined at a point $t \in \mathbb{T}^\kappa$ by

$$f^\Delta(t) := \lim_{\substack{s \to t \\ s \neq \sigma(t)}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \tag{1.2}$$

was first introduced by Hilger [22] to unify discrete and continuous analyses. It follows from the definition of the operator σ that

$$\sigma(t) = \begin{cases} t & \text{if } \mathbb{T} = \mathbb{R} \\ t+1 & \text{if } \mathbb{T} = \mathbb{Z} \\ qt & \text{if } \mathbb{T} = \overline{q^{\mathbb{Z}}} \\ t+h & \text{if } \mathbb{T} = h\mathbb{Z}, \end{cases} \tag{1.3}$$

where $\overline{q^{\mathbb{Z}}} = \{q^k : k \in \mathbb{Z} \text{ and } q > 1\} \cup \{0\}$ and $h\mathbb{Z} = \{hn : n \in \mathbb{Z} \text{ and } h > 0\}$. Hence, the delta derivative $f^{\Delta}(t)$ turns into ordinary derivative $f'(t)$ if $\mathbb{T} = \mathbb{R}$ and it becomes the forward h-difference operator $\Delta_h f(t) := \frac{1}{h}[f(t+h) - f(t)]$ whenever $\mathbb{T} = h\mathbb{Z}$ (i.e. $f^{\Delta}(t) = f(t+1) - f(t) = \Delta f(t)$ if $h = 1$). For the time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ we have $f^{\Delta}(t) = D_q f(t)$, where

$$D_q f(t) = \frac{f(qt) - f(t)}{(q-1)t}. \tag{1.4}$$

It follows from (1.2) and (1.3) that dynamic equations on time scales turn into difference equations when the time scale is chosen as the set of integers, and they become differential equations when the time scale coincides with the set of reals. Moreover, q-difference, h-difference equations, used in the discretization of differential equations, are all particular cases of dynamic equations on time scales. Since there are many time scales other than the sets of reals and integers, analysis on time scales provides a more general theory which enables us to see similarities and differences between the analyses on discrete and continuous time domains.

Throughout the paper, we denote by $[a, b]_{\mathbb{T}}$ the closed time scale interval $[a, b] \cap \mathbb{T}$. The other time scale intervals $[a, b)_{\mathbb{T}}$, $(a, b]_{\mathbb{T}}$, and $(a, b)_{\mathbb{T}}$ are defined similarly. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right dense points and its left sided limits exists (finite) at left dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T})$. It is known by [23, Theorem 1.60] that the forward jump operator defined by (1.1) is an rd-continuous. By [23, Theorem 1.65] it is concluded that every rd-continuous function is bounded on a compact interval. Note that continuity implies rd-continuity. Every rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ has an anti-derivative

$$F(t) = \int_{t_0}^{t} f(t) \Delta t.$$

That is, $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$ (see [24, Theorem 1.74]). For an excellent review on Δ-derivative and Δ-Riemann integral we refer the reader to [23].

Hereafter, we give some basic definitions and theorems that will be used in further sections.

Definition 1. A function $h : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* provided $1 + \mu(t)h(t) \neq 0$ for all $t \in \mathbb{T}^{\kappa}$, where $\mu(t) = \sigma(t) - t$. The set of all regressive rd-continuous functions $\varphi : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by $\mathcal{R}^+ = \{h \in \mathcal{R} : 1 + \mu(t)\varphi(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $\varphi \in \mathcal{R}$. The *exponential function* on $\mathbb{T}$ is defined by

$$e_{\varphi}(t, s) = \exp\left(\int_{s}^{t} \zeta_{\mu(r)}(\varphi(r)) \Delta r\right) \tag{1.5}$$

where $\zeta_{\mu(s)}$ is the cylinder transformation given by

$$\zeta_{\mu(r)}(\varphi(r)) := \begin{cases} \dfrac{1}{\mu(r)} \mathrm{Log}(1 + \mu(r)\varphi(r)) & \text{if } \mu(r) > 0 \\ \varphi(r) & \text{if } \mu(r) = 0. \end{cases} \tag{1.6}$$

It is well known that (see [25, Theorem 14]) if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\Delta} = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following results:

Lemma 2 ([23, Theorem 2.36]). *Let $p, q \in \mathcal{R}$. Then*

i. $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
ii. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
iii. $\dfrac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$ where $\ominus p(t) = -\dfrac{p(t)}{1+\mu(t)p(t)}$;
iv. $e_p(t, s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
v. $e_p(t, s)e_p(s, r) = e_p(t, r)$;
vi. $\left(\dfrac{1}{e_p(\cdot, s)}\right)^{\Delta} = -\dfrac{p(t)}{e_p^{\sigma}(\cdot, s)}$.

Lemma 3 ([26]). *For a nonnegative φ with $-\varphi \in \mathcal{R}^+$, we have the inequalities*

$$1 - \int_{s}^{t} \varphi(u) \leq e_{-\varphi}(t, s) \leq \exp\left\{-\int_{s}^{t} \varphi(u)\right\} \quad \text{for all } t \geq s.$$

If φ is rd-continuous and nonnegative, then

$$1 + \int_{s}^{t} \varphi(u) \leq e_{\varphi}(t, s) \leq \exp\left\{\int_{s}^{t} \varphi(u)\right\} \quad \text{for all } t \geq s.$$

Remark 1 (*[27, Remark 2.12]*)**.** If $\lambda \in \mathcal{R}^+$ and $\lambda(r) < 0$ for all $t \in [s, t)_{\mathbb{T}}$, then

$$0 < e_\lambda(t, s) \leq \exp \left(\int_s^t \lambda(r) \Delta r \right) < 1.$$

2. Shift operators and delay functions

2.1. Shift operators

First, we give a generalized version of shift operators (see [16,28]). A limited version of shift operators can be found in [29].

Definition 2 (*Shift Operators [16]*)**.** Let $\mathbb{T}^*$ be a non-empty subset of the time scale $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exist operators $\delta_\pm : [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties:

P.1 The functions $\delta_\pm$ are strictly increasing with respect to their second arguments, i.e., if

$$(T_0, t), (T_0, u) \in \mathcal{D}_\pm := \left\{ (s, t) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_\pm(s, t) \in \mathbb{T}^* \right\},$$

then

$$T_0 \leq t < u \quad \text{implies } \delta_\pm(T_0, t) < \delta_\pm(T_0, u),$$

P.2 If $(T_1, u), (T_2, u) \in \mathcal{D}_-$ with $T_1 < T_2$, then

$$\delta_-(T_1, u) > \delta_-(T_2, u),$$

and if $(T_1, u), (T_2, u) \in \mathcal{D}_+$ with $T_1 < T_2$, then

$$\delta_+(T_1, u) < \delta_+(T_2, u),$$

P.3 If $t \in [t_0, \infty)_{\mathbb{T}}$, then $(t, t_0) \in \mathcal{D}_+$ and $\delta_+(t, t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0, t) \in \mathcal{D}_+$ and $\delta_+(t_0, t) = t$ holds,
P.4 If $(s, t) \in \mathcal{D}_\pm$, then $(s, \delta_\pm(s, t)) \in \mathcal{D}_\mp$ and $\delta_\mp(s, \delta_\pm(s, t)) = t$,
P.5 If $(s, t) \in \mathcal{D}_\pm$ and $(u, \delta_\pm(s, t)) \in \mathcal{D}_\mp$, then $(s, \delta_\mp(u, t)) \in \mathcal{D}_\pm$ and

$$\delta_\mp(u, \delta_\pm(s, t)) = \delta_\pm(s, \delta_\mp(u, t)).$$

Then the operators δ_- and δ_+ associated with $t_0 \in \mathbb{T}^*$ (called the initial point) are said to be *backward and forward shift operators* on the set $\mathbb{T}^*$, respectively. The variable $s \in [t_0, \infty)_{\mathbb{T}}$ in $\delta_\pm(s, t)$ is called the shift size. The values $\delta_+(s, t)$ and $\delta_-(s, t)$ in $\mathbb{T}^*$ indicate s units translation of the term $t \in \mathbb{T}^*$ to the right and left, respectively. The sets $\mathcal{D}_\pm$ are the domains of the shift operators $\delta_\pm$, respectively.

Example 1 (*[16]*)**.** Let $\mathbb{T} = \mathbb{R}$ and $t_0 = 1$. The operators

$$\delta_-(s, t) = \begin{cases} t/s & \text{if } t \geq 0 \\ st & \text{if } t < 0, \end{cases} \quad \text{for } s \in [1, \infty) \tag{2.1}$$

and

$$\delta_+(s, t) = \begin{cases} st & \text{if } t \geq 0 \\ t/s & \text{if } t < 0, \end{cases} \quad \text{for } s \in [1, \infty) \tag{2.2}$$

are backward and forward shift operators (on the set $\mathbb{T}^* = \mathbb{R} - \{0\}$) associated with the initial point $t_0 = 1$. In the table below, we state different time scales with their corresponding shift operators.

$\mathbb{T}$	t_0	$\mathbb{T}^*$	$\delta_-(s, t)$	$\delta_+(s, t)$
$\mathbb{R}$	0	$\mathbb{R}$	$t - s$	$t + s$
$\mathbb{Z}$	0	$\mathbb{Z}$	$t - s$	$t + s$
$q^{\mathbb{Z}} \cup \{0\}$	1	$q^{\mathbb{Z}}$	$\frac{t}{s}$	st
$\mathbb{N}^{1/2}$	0	$\mathbb{N}^{1/2}$	$\sqrt{t^2 - s^2}$	$\sqrt{t^2 + s^2}$

The proof of the next lemma is a direct consequence of Definition 2.

Lemma 4 (*[16]*)**.** *Let δ_- and δ_+ be the shift operators associated with the initial point t_0. We have*

  i. $\delta_-(t, t) = t_0$ *for all* $t \in [t_0, \infty)_{\mathbb{T}}$,
  ii. $\delta_-(t_0, t) = t$ *for all* $t \in \mathbb{T}^*$,

iii. *If $(s, t) \in \mathcal{D}_+$, then $\delta_+(s, t) = u$ implies $\delta_-(s, u) = t$. Conversely, if $(s, u) \in \mathcal{D}_-$, then $\delta_-(s, u) = t$ implies $\delta_+(s, t) = u$.*

iv. *$\delta_+(t, \delta_-(s, t_0)) = \delta_-(s, t)$ for all $(s, t) \in \mathcal{D}(\delta_+)$ with $t \geq t_0$,*

v. *$\delta_+(u, t) = \delta_+(t, u)$ for all $(u, t) \in ([t_0, \infty)_{\mathbb{T}} \times [t_0, \infty)_{\mathbb{T}}) \cap \mathcal{D}_+$,*

vi. *$\delta_+(s, t) \in [t_0, \infty)_{\mathbb{T}}$ for all $(s, t) \in \mathcal{D}_+$ with $t \geq t_0$,*

vii. *$\delta_-(s, t) \in [t_0, \infty)_{\mathbb{T}}$ for all $(s, t) \in ([t_0, \infty)_{\mathbb{T}} \times [s, \infty)_{\mathbb{T}}) \cap \mathcal{D}_-$,*

viii. *If $\delta_+(s, .)$ is Δ-differentiable in its second variable, then $\delta_+^{\Delta t}(s, .) > 0$,*

ix. *$\delta_+(\delta_-(u, s), \delta_-(s, v)) = \delta_-(u, v)$ for all $(s, v) \in ([t_0, \infty)_{\mathbb{T}} \times [s, \infty)_{\mathbb{T}}) \cap \mathcal{D}_-$ and $(u, s) \in ([t_0, \infty)_{\mathbb{T}} \times [u, \infty)_{\mathbb{T}}) \cap \mathcal{D}_-$,*

x. *If $(s, t) \in \mathcal{D}_-$ and $\delta_-(s, t) = t_0$, then $s = t$.*

2.2. Delay functions generated by shift operators

Next, we define the delay function by means of shift operators on time scales. Delay functions generated by shift operators were first introduced in [16] to construct delay equations on time scales.

Definition 3 (*Delay Functions [16]*)**.** Let $\mathbb{T}$ be a time scale that is unbounded above and $\mathbb{T}^*$ an unbounded subset of $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exist shift operators $\delta_\pm : [t_0, \infty)_{\mathbb{T}} \times \mathbb{T}^* \rightarrow \mathbb{T}^*$ associated with t_0. Suppose that $h \in (t_0, \infty)_{\mathbb{T}}$ is a constant such that $(h, t) \in D_\pm$ for all $t \in [t_0, \infty)_{\mathbb{T}}$, the function $\delta_-(h, t)$ is differentiable with an rd-continuous derivative $\delta_-^{\Delta t}(h, t)$, and $\delta_-(h, t)$ maps $[t_0, \infty)_{\mathbb{T}}$ onto $[\delta_-(h, t_0), \infty)_{\mathbb{T}}$. Then the function $\delta_-(h, t)$ is called the delay function generated by the shift δ_- on the time scale $\mathbb{T}$.

It is obvious from P.2 in Definition 3 and (ii) of Lemma 4 that

$$\delta_-(h, t) < \delta_-(t_0, t) = t \quad \text{for all } t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.3}$$

Notice that $\delta_-(h, .)$ is strictly increasing and it is invertible. Hence, by P. 4–5

$$\delta_-^{-1}(h, t) = \delta_+(h, t).$$

Hereafter, we shall suppose that $\mathbb{T}$ is a time scale with the delay function $\delta_-(h, .) : [t_0, \infty)_{\mathbb{T}} \rightarrow [\delta_-(h, t_0), \infty)_{\mathbb{T}}$, where $t_0 \in \mathbb{T}$ is fixed. Denote by $\mathbb{T}_1$ and $\mathbb{T}_2$ the sets

$$\mathbb{T}_1 = [t_0, \infty)_{\mathbb{T}} \quad \text{and} \quad \mathbb{T}_2 = \delta_-(h, \mathbb{T}_1). \tag{2.4}$$

Evidently, $\mathbb{T}_1$ is closed in $\mathbb{R}$. By definition we have $\mathbb{T}_2 = [\delta_-(h, t_0), \infty)_{\mathbb{T}}$. Hence, $\mathbb{T}_1$ and $\mathbb{T}_2$ are both time scales. Let σ_1 and σ_2 denote the forward jumps on the time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. By ((2.3)–(2.4))

$$\mathbb{T}_1 \subset \mathbb{T}_2 \subset \mathbb{T}.$$

Thus,

$$\sigma(t) = \sigma_2(t) \quad \text{for all } t \in \mathbb{T}_2$$

and

$$\sigma(t) = \sigma_1(t) = \sigma_2(t) \quad \text{for all } t \in \mathbb{T}_1.$$

That is, σ_1 and σ_2 are the restrictions of forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ to the time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively, i.e.,

$$\sigma_1 = \sigma|_{\mathbb{T}_1} \quad \text{and} \quad \sigma_2 = \sigma|_{\mathbb{T}_2}.$$

Lemma 5 (*[16]*)**.** *The delay function $\delta_-(h, t)$ preserves the structure of the points in $\mathbb{T}_1$. That is,*

$$\sigma_1(\widehat{t}) = \widehat{t} \quad \text{implies} \quad \sigma_2(\delta_-(h, \widehat{t})) = \delta_-(h, \widehat{t})$$
$$\sigma_1(\widehat{t}) > \widehat{t} \quad \text{implies} \quad \sigma_2(\delta_-(h, \widehat{t})) > \delta_-(h, \widehat{t}).$$

Using the preceding lemma and applying the fact that $\sigma_2(u) = \sigma(u)$ for all $u \in \mathbb{T}_2$ we arrive at the following result.

Corollary 1 (*[16]*)**.** *We have*

$$\delta_-(h, \sigma_1(t)) = \sigma_2(\delta_-(h, t)) \quad \text{for all } t \in \mathbb{T}_1.$$

Thus,

$$\delta_-(h, \sigma(t)) = \sigma(\delta_-(h, t)) \quad \text{for all } t \in \mathbb{T}_1. \tag{2.5}$$

By (2.5) we have

$$\delta_-(h, \sigma(s)) = \sigma(\delta_-(h, s)) \quad \text{for all } s \in [t_0, \infty)_{\mathbb{T}}.$$

Substituting $s = \delta_+(h, t)$ we obtain

$$\delta_-(h, \sigma(\delta_+(h, t))) = \sigma(\delta_-(h, \delta_+(h, t))) = \sigma(t).$$

This and (iv) of Lemma 4 imply

$$\sigma(\delta_+(h, t)) = \delta_+(h, \sigma(t)) \quad \text{for all } t \in [\delta_-(h, t_0), \infty)_{\mathbb{T}}.$$

Example 2. In the following, we give some time scales with their shift operators:

$\mathbb{T}$	h	$\delta_-(h, t)$	$\delta_+(h, t)$
$\mathbb{R}$	$\in \mathbb{R}_+$	$t - h$	$t + h$
$\mathbb{Z}$	$\in \mathbb{Z}_+$	$t - h$	$t + h$
$q^{\mathbb{Z}} \cup \{0\}$	$\in q^{\mathbb{Z}_+}$	$\frac{t}{h}$	ht
$\mathbb{N}^{1/2}$	$\in \mathbb{Z}_+$	$\sqrt{t^2 - h^2}$	$\sqrt{t^2 + h^2}$

Example 3. There is no delay function $\delta_-(h, .) : [0, \infty)_{\widetilde{\mathbb{T}}} \to [\delta_-(h, 0), \infty)_{\mathbb{T}}$ on the time scale $\widetilde{\mathbb{T}} = (-\infty, 0] \cup [1, \infty)$.

Suppose contrary that there exists a such delay function on $\widetilde{\mathbb{T}}$. Then since 0 is right scattered in $\widetilde{\mathbb{T}}_1 := [0, \infty)_{\widetilde{\mathbb{T}}}$ the point $\delta_-(h, 0)$ must be right scattered in $\widetilde{\mathbb{T}}_2 = [\delta_-(h, 0), \infty)_{\mathbb{T}}$, i.e., $\sigma_2(\delta_-(h, 0)) > \delta_-(h, 0)$. Since $\sigma_2(t) = \sigma(t)$ for all $t \in [\delta_-(h, 0), 0)_{\mathbb{T}}$, we have

$$\sigma(\delta_-(h, 0)) = \sigma_2(\delta_-(h, 0)) > \delta_-(h, 0).$$

That is, $\delta_-(h, 0)$ must be right scattered in $\widetilde{\mathbb{T}}$. However, in $\widetilde{\mathbb{T}}$ we have $\delta_-(h, 0) < 0$, that is, $\delta_-(h, 0)$ is right dense. This leads to a contradiction.

3. Halanay type inequalities on time scales

Let $\mathbb{T}$ be a time scale that is unbounded above and $t_0 \in \mathbb{T}^*$ an element such that there exist the shift operators $\delta_\pm : [t_0, \infty) \times \mathbb{T}^* \to \mathbb{T}^*$ associated with t_0. Suppose that $h_1, h_2, \ldots, h_r \in (t_0, \infty)_{\mathbb{T}}$ are the constants with

$$t_0 = h_0 < h_1 < h_2 < \cdots < h_r$$

and that there exist delay functions $\delta_-(h_i, t)$, $i = 1, 2, \ldots, r$, on $\mathbb{T}$.

We define lower Δ-derivative $\varphi^{\Delta_-}(t)$ of a function $\varphi : \mathbb{T} \to \mathbb{R}$ on time scales as follows:

$$\varphi^{\Delta_-}(t) = \liminf_{s \to t^-} \frac{\varphi(s) - \varphi(\sigma(t))}{s - \sigma(t)}. \tag{3.1}$$

Notice that

$$\varphi^{\Delta_-}(t) = \varphi^{\Delta}(t)$$

provided that φ is Δ-differentiable at $t \in \mathbb{T}^\kappa$.

Let $f(t, u, v)$ be a continuous function for all (u, v) and $t \in [t_0, \alpha)_{\mathbb{T}}$. Hereafter, we suppose that f is monotone increasing with respect to v and non-decreasing with respect to u.

Proposition 1. *Let $g(u_1, u_2, \ldots, u_r)$ be a continuous function that is monotone increasing with respect to each of its arguments. If φ and ψ are continuous functions satisfying*

$$\varphi^{\Delta_-}(t) < f\left(t, \varphi(t), g\left(\varphi(\delta_-(h_1, t)), \varphi(\delta_-(h_2, t)), \ldots, \varphi(\delta_-(h_r, t))\right)\right),$$
$$\psi^{\Delta_-}(t) \geq f\left(t, \psi(t), g\left(\psi(\delta_-(h_1, t)), \psi(\delta_-(h_2, t)), \ldots, \psi(\delta_-(h_r, t))\right)\right),$$

for all $t \in [t_0, \alpha)_{\mathbb{T}}$ and $\varphi(s) < \psi(s)$ for all $s \in [\delta_-(h_r, t_0), t_0]_{\mathbb{T}}$, then

$$\varphi(t) < \psi(t) \quad \text{for all } t \in (t_0, \alpha)_{\mathbb{T}}, \tag{3.2}$$

where $\alpha \in (t_0, \infty)_{\mathbb{T}}$.

Proof. Suppose that (3.2) does not hold for some $t \in (t_0, \alpha)_{\mathbb{T}}$. Then the set

$$M := \{t \in (t_0, \alpha)_{\mathbb{T}} : \varphi(t) \geq \psi(t)\}$$

is non-empty. Since M is bounded below we can let $\xi := \inf M$. If ξ is left scattered (i.e. $\sigma(\rho(\xi)) = \xi$), then it follows from the definition of ξ that

$$\varphi(\rho(\xi)) < \psi(\rho(\xi)),$$
$$\varphi(\xi) \geq \psi(\xi).$$

Since $\rho(\xi)$ is right scattered, the function φ is Δ-differentiable at $\rho(\xi)$ (see [23, Theorem 1.16, (ii)]), and hence, $\varphi^{\Delta^-}(\rho(\xi)) = \varphi^{\Delta}(\rho(\xi))$. Similarly we obtain $\psi^{\Delta^-}(\rho(\xi)) = \psi^{\Delta}(\rho(\xi))$. Thus,

$$
\begin{aligned}
\varphi(\xi) &= \varphi(\sigma(\rho(\xi))) \\
&= \varphi(\rho(\xi)) + \mu(\rho(\xi))\varphi^{\Delta}(\rho(\xi)) \\
&= \varphi(\rho(\xi)) + \mu(\rho(\xi))\varphi^{\Delta^-}(\rho(\xi)) \\
&< \varphi(\rho(\xi)) + \mu(\rho(\xi))f\left(\rho(\xi), \varphi(\rho(\xi)), g\left(\varphi(\delta_-(h_1, \rho(\xi))), \varphi(\delta_-(h_2, \rho(\xi))), \ldots, \varphi(\delta_-(h_r, \rho(\xi)))\right)\right) \\
&< \psi(\rho(\xi)) + \mu(\rho(\xi))f\left(\rho(\xi), \psi(\rho(\xi)), g\left(\psi(\delta_-(h_1, \rho(\xi))), \psi(\delta_-(h_2, \rho(\xi))), \ldots, \psi(\delta_-(h_r, \rho(\xi)))\right)\right) \\
&\leq \psi(\rho(\xi)) + \mu(\rho(\xi))\psi^{\Delta^-}(\rho(\xi)) \\
&= \psi(\rho(\xi)) + \mu(\rho(\xi))\psi^{\Delta}(\rho(\xi)) \\
&= \psi(\sigma(\rho(\xi))) \\
&= \psi(\xi).
\end{aligned}
$$

This leads to a contradiction. If ξ is left dense, then we have $\xi > t_0$ and

$$\varphi(\xi) = \psi(\xi).$$

Since

$$\delta_-(h_r, \xi) < \xi \quad \text{for all } i = 1, 2, \ldots, r$$

and

$$\varphi(s) < \psi(s) \quad \text{for all } s \in [\delta_-(h_r, \xi), \xi)_{\mathbb{T}},$$

we obtain

$$g(\varphi(\delta_-(h_1, \xi)), \ldots, \varphi(\delta_-(h_r, \xi))) \leq g(\psi(\delta_-(h_1, \xi)), \ldots, \psi(\delta_-(h_r, \xi))),$$

and therefore,

$$
\begin{aligned}
\varphi^{\Delta^-}(\xi) &< f\left(\xi, \varphi(\xi), g\left(\varphi(\delta_-(h_1, \xi)), \varphi(\delta_-(h_2, \xi)), \ldots, \varphi(\delta_-(h_r, \xi))\right)\right) \\
&\leq f\left(\xi, \psi(\xi), g\left(\psi(\delta_-(h_1, \xi)), \psi(\delta_-(h_2, \xi)), \ldots, \psi(\delta_-(h_r, \xi))\right)\right) \\
&\leq \psi^{\Delta^-}(\xi).
\end{aligned}
$$

On the other hand, since

$$\frac{\varphi(s) - \varphi(\sigma(\xi))}{s - \sigma(\xi)} \geq \frac{\psi(s) - \psi(\sigma(\xi))}{s - \sigma(\xi)}$$

for all $s \in [\delta_-(h_r, \xi), \xi)_{\mathbb{T}}$ we get by (3.1) that

$$\varphi^{\Delta^-}(\xi) \geq \psi^{\Delta^-}(\xi).$$

This also leads to a contradiction and so this completes the proof. $\quad\square$

Proposition 2. *If*

$$\omega^{\Delta}(t) \leq f\left(t, \omega(t), g\left(\omega(\delta_-(h_1, t)), \omega(\delta_-(h_2, t)), \ldots, \omega(\delta_-(h_r, t))\right)\right)$$

for $t \in [s_0, \delta_+(\alpha, s_0))_{\mathbb{T}}$ and $y(t; s_0, \omega)$ is a solution of the equation

$$y^{\Delta}(t) = f\left(t, y(t), g\left(y(\delta_-(h_1, t)), y(\delta_-(h_2, t)), \ldots, y(\delta_-(h_r, t))\right)\right),$$

which coincides with ω in $[\delta_-(h_r, s_0), s_0]_{\mathbb{T}}$, then, supposing that this solution is defined in $[s_0, \delta_+(\alpha, s_0))_{\mathbb{T}}$, it follows that $\omega(t) \leq y(t; s_0, \omega)$ for $t \in [s_0, \delta_+(\alpha, s_0))_{\mathbb{T}}$.

Proof. Let ε_n be a sequence of positive numbers tending monotonically to zero, and y_n be a solution of the equation

$$y^{\Delta}(t) = f\left(t, y(t), g\left(y(\delta_-(h_1, t)), y(\delta_-(h_2, t)), \ldots, y(\delta_-(h_r, t))\right)\right) + \varepsilon_n,$$

which in $[\delta_-(h_r, s_0), s_0]_{\mathbb{T}}$ coincides with $\omega + \varepsilon_n$. On the basis of the preceding proposition, we have

$$y_{n+1}(t) < y_n(t)$$

and

$$\lim_{n \to \infty} y_n(t) = y(t; s_0, \omega)$$

for all $t \in [s_0, \delta_+(\alpha, s_0))_{\mathbb{T}}$. On the basis of Proposition 1 we have $\omega(t) < y_n(t)$ for $t \in [s_0, \delta_+(\alpha, s_0))_{\mathbb{T}}$, and hence, $\omega(t) \leq y(t; s_0, \omega)$. The proof is complete. $\quad\square$

Hereafter, we will denote by $\widetilde{\mu}$ the function defined by

$$\widetilde{\mu}(t) := \sup_{s \in [\delta_-(h_r, t_0), t]_{\mathbb{T}}} \mu(s) \tag{3.3}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$. It is obvious that the sets $\mathbb{R}, \mathbb{Z}, \overline{q^{\mathbb{Z}}} = \{q^n : n \in \mathbb{Z} \text{ and } q > 1\} \cup \{0\}, h\mathbb{Z} = \{hn : n \in \mathbb{Z} \text{ and } h > 0\}$ are the examples of time scales on which $\widetilde{\mu} = \mu$.

Theorem 2. *Let x be a function satisfying the inequality*

$$x^{\Delta}(t) \leq -p(t)x(t) + \sum_{i=0}^{r} q_i(t)x^{\ell}(\delta_-(h_i, t)), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{3.4}$$

where $\ell \in (0, 1]$ is a constant, p and q_i, $i = 0, 1, \ldots, r$, are continuous and bounded functions satisfying $1 - \widetilde{\mu}(t)p(t) \geq 0$; $q_i(t) \geq 0$, $i = 0, 1, \ldots, r - 1$; $q_r(t) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Suppose that

$$p(t) - \sum_{i=0}^{r} q_i(t) > 0 \quad \text{for all } t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.5}$$

Then there exist a positively regressive function $\lambda : [t_0, \infty)_{\mathbb{T}} \to (-\infty, 0)$ and $K_0 > 1$ such that

$$x(t) \leq K_0 e_{\lambda}(t, t_0) \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.6}$$

Proof. Consider the delay dynamic equation

$$y^{\Delta}(t) = -p(t)y(t) + \sum_{i=0}^{r} q_i(t)y^{\ell}(\delta_-(h_i, t)), \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.7}$$

We look for a solution of Eq. (3.7) in the form $e_{\lambda}(t, t_0)$, where $\lambda : \mathbb{T} \to (-\infty, 0)$ is positively regressive (i.e. $1 + \mu(t)\lambda(t) > 0$) and rd-continuous. First note that

$$e_{\lambda}^{\Delta}(t, t_0) = \lambda(t)e_{\lambda}(t, t_0).$$

For a given $K > 1$, the function $Ke_{\lambda}(t, t_0)$ is a solution of (3.7) if and only if λ is a root of the characteristic polynomial $P(t, \lambda)$ defined by

$$P(t, \lambda) := (\lambda + p(t)) \, e_{\lambda}(t, \delta_-(h_r, t))e_{\lambda}^{1-\ell}(\delta_-(h_r, t), t_0) - K^{\ell-1} \sum_{i=0}^{r} q_i(t)e_{\lambda}^{\ell}(\delta_-(h_i, t), \delta_-(h_r, t)). \tag{3.8}$$

For each fixed $t \in [t_0, \infty)_{\mathbb{T}}$ define the set

$$S(t) := \{k \in (-\infty, 0) : 1 + \widetilde{\mu}(t)k > 0\}. \tag{3.9}$$

It follows from Lemma 3 that if k is a scalar in $S(t)$, then $0 < 1 + \widetilde{\mu}(t)k \leq 1 + \mu(u)k$ for all $u \in [\delta_-(h_r, t_0), t]_{\mathbb{T}}$ and

$$0 < e_k(\tau, s) \leq \exp(k(\tau - s)), \tag{3.10}$$

for all $\tau \in [\delta_-(h_r, t_0), t]_{\mathbb{T}}$ with $\tau \geq s$. It is obvious from (1.5) and (1.6) that for each fixed $t \in [t_0, \infty)_{\mathbb{T}}$ the function $P(t, k)$ is continuous with respect to k in $S(t)$. Since $e_0(t, t_0) = 1$ we have

$$P(t, 0) = p(t) - K^{\ell-1} \sum_{i=0}^{r} q_i(t) > 0. \tag{3.11}$$

Let $t \in [t_0, \infty)_{\mathbb{T}}$ be fixed. If the interval $[\delta_-(h_r, t_0), t]_{\mathbb{T}}$ has no any right scattered points, then $\widetilde{\mu}(t) = 0$ and $S(t) = (-\infty, 0)$. By (3.10), we get

$$\lim_{k \to -\infty} e_k(t, s) = 0,$$

and hence,

$$\lim_{k \to -\infty} P(t, k) = -K^{\ell-1}q_r(t) < 0.$$

If the interval $[\delta_-(h_r, t_0), t]_{\mathbb{T}}$ has some right scattered points (i.e. if $\widetilde{\mu}(t) > 0$), then we have $S(t) = \left(-\frac{1}{\widetilde{\mu}(t)}, 0\right)$. For all

$k \in \left(-\frac{1}{\widetilde{\mu}(t)}, 0 \right)$ we have $e_k(t, s) > 0$. Since $1 - \widetilde{\mu}(t)p(t) \geq 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$, we obtain

$$\lim_{k \to -\frac{1}{\widetilde{\mu}(t)}^+} P(t, k) = \left(-\frac{1}{\widetilde{\mu}(t)} + p(t) \right) \lim_{k \to -\frac{1}{\widetilde{\mu}(t)}^+} \left[e_k(t, \delta_-(h_r, t)) e_k^{1-\ell}(\delta_-(h_r, t), t_0) \right]$$

$$- K^{\ell-1} \sum_{i=0}^{r-1} q_i(t) \lim_{k \to -\frac{1}{\widetilde{\mu}(t)}^+} e_k^\ell(\delta_-(h_i, t), \delta_-(h_r, t)) - K^{\ell-1} q_r(t)$$

$$< -K^{\ell-1} q_r(t) < 0.$$

Therefore, for each fixed $t \in [t_0, \infty)_{\mathbb{T}}$, we obtain

$$0 > -K^{\ell-1}q_r(t) \geq \begin{cases} \displaystyle\lim_{k \to -\frac{1}{\widetilde{\mu}(t)}^+} P(t, k) & \text{if } \widetilde{\mu}(t) > 0 \\ \displaystyle\lim_{k \to -\infty} P(t, k) & \text{if } \widetilde{\mu}(t) = 0. \end{cases} \tag{3.12}$$

It follows from the continuity of P in k and ((3.11)–(3.12)) that for each fixed $t \in [t_0, \infty)_{\mathbb{T}}$, there exists a largest element k_0 of the set $S(t)$ such that

$$P(t, k_0) = 0.$$

Using all these largest elements we can construct a positively regressive function $\lambda : [\delta_-(h_r, t_0), \infty)_{\mathbb{T}} \to (-\infty, 0)$ by

$$\lambda(t) := \max \{k \in S(t) : P(t, k) = 0\} \tag{3.13}$$

so that for a given $K > 1$, $y(t) = Ke_\lambda(t, t_0)$ is a solution to (3.7).

If $y(t)$ be a solution of (3.7), $x(t)$ satisfies (3.4), and $x(t) \leq y(t)$ for all $t \in [\delta_-(h_r, t_0), t_0]_{\mathbb{T}}$, then by Proposition 2 the inequality $x(t) \leq y(t)$ holds for all $t \in [t_0, \infty)_{\mathbb{T}}$. For a given $K > 1$, we have

$$\inf_{t \in [\delta_-(h_r, t_0), t_0]_{\mathbb{T}}} Ke_\lambda(t, t_0) = K,$$

hence, by choosing a $K_0 > 1$ with

$$K_0 > \sup_{t \in [\delta_-(h_r, t_0), t_0]_{\mathbb{T}}} x(t),$$

we get

$$x(t) < K_0 e_\lambda(t, t_0) \quad \text{for all } t \in [\delta_-(h_r, t_0), t_0]_{\mathbb{T}}.$$

It follows on the basis of Proposition 2 that the inequality

$$x(t) \leq K_0 e_\lambda(t, t_0)$$

holds for all $t \in [t_0, \infty)_{\mathbb{T}}$. This completes the proof. □

In next two examples, we apply Theorem 2 to the time scales $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = q^{\mathbb{N}}$ to derive some results for difference and q-difference inequalities.

Example 4. Let $\mathbb{T} = \mathbb{Z}$; $t_0 = 0$; $\delta_-(h_i, t) = t - h_i$, $h_i \in \mathbb{N}$, $i = 1, 2, \ldots, r - 1$; $h_r \in \mathbb{Z}^+$, and $0 = h_0 < h_1 < \cdots < h_r$. Assume that p and $q_i \geq 0$, $i = 0, 1, 2, \ldots, r$, are the scalars satisfying $q_r > 0$ and

$$\sum_{i=0}^{r} q_i < p \leq 1.$$

Then the Eq. (3.7) becomes

$$\Delta y(t) = -py(t) + \sum_{i=0}^{r} q_i y^\ell(t - h_i), \quad t \in \{0, 1, \ldots\}. \tag{3.14}$$

The characteristic polynomial and the set $S(t)$ given by (3.8) and (3.9) turn into

$$P(t, \lambda) = (\lambda + p)(1 + \lambda)^{h_r}(1 + \lambda)^{(1-\ell)(t-h_r)} - K^{\ell-1} \sum_{i=0}^{r} q_i(1 + \lambda)^{\ell(h_r - h_i)}$$

and

$$S(t) = (-1, 0) \quad \text{for all } t \in \{0, 1, \ldots\},$$

respectively. Let $\{x(t)\}$, $t \in [-h_r, \infty)_{\mathbb{Z}}$ be a sequence satisfying the inequality

$$\Delta x(t) \le -px(t) + \sum_{i=0}^{r} q_i x^\ell (t - h_i), \quad t \in \{0, 1, \ldots\}.$$

Then by Theorem 2, we conclude that there exists a constant $K_0 > 1$ such that

$$x(t) < K_0 \prod_{s=0}^{t-1} (1 + \lambda_0(s)), \quad t \in \{0, 1, \ldots\},$$

where $\lambda_0 : \mathbb{Z} \to (-1, 0)$ is a positively regressive function defined by

$$\lambda_0(t) = \max \left\{ v \in (-1, 0) : (v + p)(1 + v)^{h_r}(1 + v)^{(1-\ell)(t-h_r)} - K^{\ell-1} \sum_{i=0}^{r} q_i (1 + v)^{\ell(h_r - h_i)} = 0 \right\}.$$

Remark 2. Example 4 shows that the result in [8, Theorem 2.1] and [6, Theorem 2.1] are the particular cases of Theorem 2 when $\mathbb{T} = \mathbb{Z}$. Moreover, unlike the ones in [8, Theorem 2.1] and [6, Theorem 2.1], the coefficients p and q_i, $i = 0, 1, \ldots, r$, of the dynamic inequality considered in Theorem 2 are allowed to depend on the parameter t. Hence, even for the particular case $\mathbb{T} = \mathbb{Z}$ we have a more general result.

Example 5. Let $\mathbb{T} = q^{\mathbb{N}} := \{q^n : n \in \mathbb{N} \text{ and } q > 1\}$, $t_0 = 1$, $\delta_-(h_i, t) = t/h_i$, where $h_i \in q^{\mathbb{N}}$, $1 = h_0 < h_1 < \cdots < h_r$. Let x be a function satisfying the inequality

$$D_q x(t) \le -p(t)x(t) + \sum_{i=0}^{r} \zeta_i(t) x^\ell \left(\frac{t}{h_i} \right), \quad t \in q^{\mathbb{N}},$$

where $D_q x(t)$ is defined as in (1.4). Let

$$q^{\mathbb{Z}} := \left\{ q^n : n \in \mathbb{Z} \text{ and } q > 1 \right\}.$$

Suppose that p and ζ_i, $i = 0, 1, \ldots, r$, are continuous and bounded functions satisfying $1 - p(t)(q-1)t \ge 0$; $\zeta_i(t) \ge 0$, $i = 1, \ldots, r-1$; $\zeta_r(t) > 0$, and

$$p(t) - \sum_{i=0}^{r} \zeta_i(t) > 0$$

for all $t \in [1, \infty) \cap q^{\mathbb{Z}}$. Then there exists a constant $K_0 > 1$ such that

$$x(t) \le K_0 \prod_{s \in [1,t) \cap q^{\mathbb{N}}} (1 + \lambda(s)(q-1)s) \quad \text{for all } t \in q^{\mathbb{N}},$$

in which λ denotes the function defined by

$$\lambda(t) := \max \{ k \in (-1/(q-1)t, 0) : P(t, k) = 0 \}, \quad t \in q^{\mathbb{N}},$$

where

$$P(t, k) := (k + p(t)) \prod_{s \in \left[\frac{t}{h_r}, t \right) \cap q^{\mathbb{Z}}} (1 + k(q-1)s) A(t, k) - K^{\ell-1} \sum_{i=0}^{r} \zeta_i(t) \prod_{s \in \left[\frac{t}{h_r}, \frac{t}{h_i} \right) \cap q^{\mathbb{Z}}} (1 + k(q-1)s)^\ell,$$

where

$$A(t, k) = \begin{cases} \displaystyle\prod_{s \in \left[1, \frac{t}{h_r} \right) \cap q^{\mathbb{Z}}} (1 + k(q-1)s)^{1-\ell} & \text{if } t > h_r \\ \displaystyle\prod_{s \in \left[\frac{t}{h_r}, 1 \right) \cap q^{\mathbb{Z}}} (1 + k(q-1)s)^{\ell-1} & \text{if } t < h_r. \end{cases}$$

Theorem 3. *Let $\tau \in [t_0, \infty)_{\mathbb{T}}$ be a constant such that there exists a delay function $\delta_-(\tau, t)$ on $\mathbb{T}$. Let x be a function satisfying the inequality*

$$x^{\Delta}(t) \le -p(t)x(t) + q(t) \sup_{s \in [\delta_-(\tau,t),t]} x^\ell(s), \quad t \in [t_0, \infty)_{\mathbb{T}},$$

where $\ell \in (0, 1]$ is a constant. Suppose that p and q are the continuous and bounded functions satisfying $p(t) > q(t) > 0$ and

$1 - \widetilde{\mu}(t)p(t) \geq 0$ *for all* $t \in [t_0, \infty)_{\mathbb{T}}$*. Then there exists a constant* $M_0 > 0$ *such that every solution x to Eq.* (4.1) *satisfies*

$$x(t) \leq M_0 e_{\widetilde{\lambda}}(t, t_0),$$

where $\widetilde{\lambda}$ *is a positively regressive function chosen as in* (3.16)*.*

Proof. We proceed as we did in the proof of Theorem 2. Consider the dynamic equation

$$y^{\Delta}(t) = -py(t) + q \sup_{s \in [\delta_-(\tau, t), t]} y^{\ell}(s), \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.15}$$

For a given $M > 1$, $Me_{\lambda}(t, t_0)$ is a solution of (3.7) if and only if λ is a root of the characteristic polynomial $\widetilde{P}(t, \lambda)$ defined by

$$\widetilde{P}(t, \lambda) := (\lambda + p(t)) e_{\lambda}(t, t_0) - M^{\ell} q(t) \sup_{s \in [\delta_-(\tau, t), t]} e_{\lambda}^{\ell}(s, t_0).$$

For each fixed $t \in [t_0, \infty)_{\mathbb{T}}$ define the set

$$S(t) := \{k \in (-\infty, 0) : 1 + \widetilde{\mu}(t)k > 0\}.$$

It is obvious that for each fixed $t \in [t_0, \infty)_{\mathbb{T}}$ and for all $k \in S(t)$ we have

$$\widetilde{P}(t, k) = (k + p(t)) e_k(t, t_0) - M^{\ell} q(t) e_k^{\ell}(\delta_-(\tau, t), t_0).$$

As we did in the proof of Theorem 2, one may easily show that for each $t \in [t_0, \infty)_{\mathbb{T}}$, there exists a largest element of $S(t)$ such that $P(t, k) = 0$. Using these largest elements we can define a positively regressive function $\widetilde{\lambda} : [\delta_-(h_r, t_0), \infty)_{\mathbb{T}} \to (-\infty, 0)$ by

$$\widetilde{\lambda}(t) := \max \left\{k \in S(t) : \widetilde{P}(t, k) = 0\right\} \tag{3.16}$$

so that for a given $M > 1 y(t) = Me_{\widetilde{\lambda}}(t, t_0)$ is a solution to (3.15). The rest of the proof can be done similar to that of Theorem 2. □

Remark 3. Notice that Theorem 3 gives Lemma 1 in the particular case when $\mathbb{T} = \mathbb{R}$ and $\ell = 1$. Moreover, since there is no nonnegativity condition on the function x, Theorem 3 provides not only a generalization but also a relaxation of Theorem 1. Similar, relaxation is valid also for the discrete case (see [9, Theorem 2.1]).

We finalize this section by giving a result for functions satisfying the dynamic inequality

$$x^{\Delta}(t) \leq -p(t)x(t) + \prod_{i=0}^{r} \beta_i(t)x^{\alpha_i}(\delta_-(h_i, t)), \tag{3.17}$$

where $\alpha_i \in (0, \infty)$, $i = 0, 1, \ldots, r$, are the scalars with $\sum_{i=0}^{r} \alpha_i = 1$. Let the characteristic polynomial $Q(t, k)$ and the set $S(t)$ be defined by

$$Q(t, k) := (\lambda + p) e_{\lambda}(t, t_0) - \prod_{i=0}^{r} \beta_i e_{\lambda}^{\alpha_i}(\delta_-(h_i, t), t_0)$$

and (3.9), respectively. Applying the similar procedure to that used in the proof of Theorem 2 we arrive at the next result.

Theorem 4. *Let* x *be a* Δ*-differentiable function satisfying* (3.17)*, where* $\alpha_i \in (0, \infty)$*,* $i = 0, 1, \ldots, r$*, are the scalars with* $\sum_{i=0}^{r} \alpha_i = 1$*;* p *and* β_i*,* $i = 0, 1, \ldots, r$*, are continuous functions with the property that* $1 - \widetilde{\mu}(t)p(t) \geq 0$*;* $\beta_i(t) > 0$*,* $i = 0, 1, \ldots, r$*, for all* $t \in [t_0, \infty)_{\mathbb{T}}$*. Suppose that*

$$p(t) - \prod_{i=0}^{r} \beta_i(t) > 0$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$*. Then there exists a constant* $L_0 > 0$ *such that*

$$x(t) \leq L_0 e_{\gamma}(t, t_0) \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$

where $\gamma : [t_0, \infty)_{\mathbb{T}} \to (-\infty, 0)$ *is a positively regressive function given by*

$$\gamma(t) := \max \{k \in S(t) : Q(t, k) = 0\}. \tag{3.18}$$

Note that Theorem 4 gives [8, Theorem 2.2] in the particular case when $\mathbb{T} = \mathbb{Z}$, $t_0 = 0$, $\delta_-(h_i, t) = t - h_i$, $i = 0, 1, 2, \ldots, r$.

4. Global stability of nonlinear dynamic equations

In this section, by means of Halanay type inequalities we gave in the previous section, we propose some sufficient conditions guaranteeing global stability of nonlinear dynamic equations in the form

$$x^{\Delta}(t) = -p(t)x(t) + F(t, x(t), x(\delta_-(h_1, t)), \ldots, x(\delta_-(h_r, t))) \tag{4.1}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$.

Theorem 5. *Let p and q_i, $i = 0, 1, \ldots, r$, be continuous and bounded functions satisfying $1 - \widetilde{\mu}(t)p(t) > 0$; $q_i(t) \geq 0$, $i = 0, 1, \ldots, r$; $q_r(t) > 0$ and*

$$p(t) - \sum_{i=0}^{r} q_i(t) > 0$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. Let $\ell \in (0, 1]$ be a constant. Assume that there exist scalars $h_i \in [t_0, \infty)_{\mathbb{T}}$, $i = 0, 1, \ldots, r$, such that $h_0 = t_0$, $\delta_-(h_i, t)$, $i = 1, \ldots, r$, are delay functions on $\mathbb{T}$, and

$$|F(t, x(t), x(\delta_-(h_1, t)), \ldots, x(\delta_-(h_r, t)))| \leq \sum_{i=0}^{r} q_i(t) \, |x(\delta_-(h_i, t))|^{\ell} \tag{4.2}$$

for all $(t, x(t), x(\delta_-(h_1, t)), \ldots, x(\delta_-(h_r, t))) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^{r+1}$. Then there exists a constant $M_0 > 1$ such that every solution x to Eq. (4.1) satisfies

$$|x(t)| \leq M_0 e_{\lambda}(t, t_0),$$

where λ is a positively regressive function chosen as in (3.13).

Proof. Let

$$\xi := \ominus(-p) = \frac{p}{1 - \mu p}.$$

Multiplying both sides of Eq. (4.1) by $e_{\xi}(t, t_0)$ and integrating the resulting equation from t_0 to t we get that

$$x(t) = x_0 e_{\ominus \xi}(t, t_0) + \int_{t_0}^{t} F(s, x(s), x(\delta_-(h_1, s)), \ldots, x(\delta_-(h_r, s))) e_{\ominus \xi}(t, \sigma(s)) \Delta s. \tag{4.3}$$

It is straightforward to show that a solution $x(t)$ to Eq. (4.3) satisfies (4.1). This means every solution of Eq. (4.1) can be rewritten in the form of (4.3). By using (4.2) we obtain

$$|x(t)| \leq |x_0| \, e_{\ominus \xi}(t, t_0) + \int_{t_0}^{t} \sum_{i=0}^{r} q_i(s) \, |x(\delta_-(h_i, s))|^{\ell} \, e_{\ominus \xi}(t, \sigma(s)) \Delta s.$$

Let the function y be defined as follows:

$$y(t) = |x(t)| \quad \text{for } t \in [\delta_-(h_r, t_0), t_0]_{\mathbb{T}}$$

and

$$y(t) = |x_0| \, e_{\ominus \xi}(t, t_0) + \int_{t_0}^{t} \sum_{i=0}^{r} q_i(s) \, |x(\delta_-(h_i, s))|^{\ell} \, e_{\ominus \xi}(t, \sigma(s)) \Delta s \quad \text{for } [t_0, \infty)_{\mathbb{T}}.$$

Then we have $|x(t)| \leq y(t)$ for all $t \in [\delta_-(h_r, t_0), \infty)_{\mathbb{T}}$. By [23, Theorem 1.117] we get that

$$y^{\Delta}(t) = -p(t) \left(|x_0| \, e_{\ominus \xi}(t, t_0) + \int_{t_0}^{t} \sum_{i=0}^{r} q_i(s) \, |x(\delta_-(h_i, s))|^{\ell} \, e_{\ominus \xi}(t, \sigma(s)) \Delta s \right) + \sum_{i=0}^{r} q_i(t) \, |x(\delta_-(h_i, t))|^{\ell}$$

$$= -p(t)y(t) + \sum_{i=0}^{r} q_i(t) \, |x(\delta_-(h_i, t))|^{\ell}$$

$$\leq -p(t)y(t) + \sum_{i=0}^{r} q_i(t) y^{\ell}(\delta_-(h_i, t))$$

for all $[t_0, \infty)_{\mathbb{T}}$. Therefore, it follows from Theorem 2 that there exists a constant $M_0 > 1$ such that

$$|x(t)| \leq M_0 e_{\lambda}(t, t_0) \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$

where $\lambda : [t_0, \infty)_{\mathbb{T}} \to (-\infty, 0)$ is a positively regressive function defined by (3.13). The proof is complete. $\quad \square$

Corollary 2. *Let x be a function satisfying the inequality*

$$x^{\Delta}(t) \leq -p(t)x(t) + q(t) \max_{i=0,1,\dots,r} \left\{ x^{\ell}(\delta_-(h_i, t)) \right\}, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.4}$$

where $\ell \in (0, 1]$ is a constant. Suppose that p and q are continuous and bounded functions satisfying $1 - \widetilde{\mu}(t)p(t) > 0$ and $p(t) > q(t) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Then, there exists a constant $M_0 > 1$ such that every solution x to Eq. (4.1) satisfies

$$|x(t)| \leq M_0 e_\lambda(t, t_0),$$

where λ is a positively regressive function chosen as in (3.13).

Similar to that of Theorem 5 one may give a proof of the following result by using Theorem 4 instead of Theorem 2.

Theorem 6. *Let p and β_i, $i = 0, 1, \dots, r$, are continuous functions satisfying $1 - \widetilde{\mu}(t)p(t) > 0$, $\beta_i(t) > 0$, $i = 0, 1, \dots, r$, for all $t \in [t_0, \infty)_{\mathbb{T}}$. Assume that $\alpha_i \in (0, \infty)$, $h_i \in [t_0, \infty)_{\mathbb{T}}$, $i = 0, 1, \dots, r$, are the scalars such that $\sum_{i=0}^{r} \alpha_i = 1$, $h_0 = t_0$, $\delta_-(h_i, t)$, $i = 1, \dots, r$, are the delay functions on $\mathbb{T}$. If*

$$p(t) - \prod_{i=0}^{r} \beta_i(t) > 0$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$ and

$$|F(t, x(t), x(\delta_-(h_1, t)), \dots, x(\delta_-(h_r, t)))| \leq \prod_{i=0}^{r} \beta_i \, |x(\delta_-(h_i, t))|^{\alpha_i}$$

for all $(t, x(t), x(\delta_-(h_1, t)), \dots, x(\delta_-(h_r, t))) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^{r+1}$. Then there exists a constant $N_0 > 1$ such that

$$x(t) \leq N_0 e_\gamma(t, t_0) \quad for \ t \in [t_0, \infty)_{\mathbb{T}},$$

where $\gamma : [t_0, \infty)_{\mathbb{T}} \to (-\infty, 0)$ is a positively regressive function given by (3.18).

Remark 4. In the case when $\mathbb{T} = \mathbb{Z}$, $p(t) = p$ and $q(t) = q$, Theorems 5 and 6 gives [8, Theorem 3.1] and [8, Theorem 3.2], respectively.

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