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Oscillation Criteria for Fourth-Order Nonlinear Dynamic Equations

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Abstract

Some oscillatory criteria for fourth order difference and differential equations are generalized to arbitrary time scales.

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1 Introduction

This paper is concerned with the oscillatory behavior of fourth-order nonlinear dynamic equations

$$(p(t)(x^{\Delta^2})^{\alpha})^{\Delta^2}(t) + q(t)f(x^{\sigma})(t) = 0, \ t \in \mathbb{T} \tag{1}$$

where α is the ratio of two positive odd integers, $p, q \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}^+)$, and $f \in \mathrm{C}(\mathbb{R}, \mathbb{R})$ such that $xf(x) > 0$ and $f'(x) \geq 0$ for $x \neq 0$.

A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers. The delta-derivative f^Δ for a function f defined on $\mathbb{T}$ turns out to be $f^\Delta = f'$ (the usual derivative) if $\mathbb{T} = \mathbb{R}$ and $f^\Delta = \Delta f$ (the usual forward difference operator) if $\mathbb{T} = \mathbb{Z}$. Here $\sigma : \mathbb{T} \mapsto \mathbb{T}$ is the forward jump operator which gives the next point in $\mathbb{T}$. The study of dynamic equations on time scales is a fairly new topic, and work in this area is rapidly growing. It is introduced well in the fundamental texts by M. Bohner and A. Peterson in [8, 9]. For recent contributions concerning the oscillation of differential, difference and dynamic equations, see the books [2, 3, 4, 5, 8, 9] and the papers [1, 10, 11, 12, 13, 15, 16].

The main purpose of this paper is to pursue a systematic study for the oscillation of equation (1). For that reason, we assume that

$$\int^\infty p^{-\frac{1}{\alpha}}(t)\Delta t = \infty \tag{2}$$

and there exists a strictly increasing function $\beta : \mathbb{T} \to \mathbb{T}$ such that $\beta^2(\mathbb{T}) := \beta(\beta(\mathbb{T}))$ is a time scale, $t < \beta(t)$ for all $t \in \mathbb{T}$ and β^2 is delta-differentiable. In Section 2, we prove a crucial lemma and give some preliminary results. In Section 3, we obtain the oscillation criteria for (1) as well as for special cases of (1) depending on $\alpha \geq 1$ and $\alpha < 1$. Finally, we establish the oscillation criteria for some delay equations in the last section.

Throughout we assume that $\mathbb{T}$ is an unbounded time scale. For convenience of notation, we let $[t_0, \infty)_\mathbb{T} = [t_0, \infty) \cap \mathbb{T},\ t_0 \in \mathbb{T}$, and $x^{\Delta^2} = x^{\Delta\Delta}$. By $\mathrm{C}(M, N)$ ($\mathrm{C_{rd}}(M, N)$) we mean the set of all continuous (right-dense continuous) functions defined on the set M to the set N. We denote $\tau = \beta^2$. We introduce the operators $L_i, i = 0, 1, 2, 3, 4$, as follows:

$$L_0x = x,\ L_1x = (L_0x)^\Delta,\ L_2x = p\{(L_1x)^\Delta\}^\alpha,\ L_3x = (L_2x)^\Delta,\ L_4x = (L_3x)^\Delta. \tag{3}$$

We recall that a solution of equation (1) is said to be *oscillatory* on $[t_0, \infty)_\mathbb{T}$ in case it is neither eventually positive nor eventually negative. Otherwise, it is said to be *nonoscillatory.* Equation (1) is said to be *oscillatory* in case all of its solutions are oscillatory.

2 Preliminaries

In this section, we first discuss possible sign conditions for the operators defined in (3) in case a solution of (1) is eventually positive. Then we obtain a crucial lemma. We finish this section with some preliminary results.

If x is an eventually positive solution of (1), then $L_4x(t) < 0$ eventually. Since (2) holds, it follows that $L_ix(t),\ i = 1, 2, 3$, are eventually of one sign. There are eight different sign combinations for these functions. It is easy to show that it is not possible that $L_ix(t) > 0$, $L_{i+1}x(t) < 0$, $L_{i+2}x(t) < 0$ and $L_ix(t) < 0$, $L_{i+1}x(t) > 0$, $L_{i+2}x(t) > 0$ for $i \in \{0, 1, 2\}$, see [7]. There are only two possibilities left, namely

(I) $L_0x(t) > 0$, $L_1x(t) > 0$, $L_2x(t) > 0$, $L_3x(t) > 0$, and $L_4x(t) < 0$ for $t \geq t_0$;

(II) $L_0x(t) > 0$, $L_1x(t) > 0$, $L_2x(t) < 0$, $L_3x(t) > 0$, and $L_4x(t) < 0$ for $t \geq t_0$, where t_0 is sufficiently large enough.

Case (I). Suppose $L_0x(t) > 0$, $L_1x(t) > 0$, $L_2x(t) > 0$, $L_3x(t) > 0$, and $L_4x(t) < 0$ for $t \geq t_0$. Since $L_3x(t) > 0$ is decreasing for $t \geq t_0$, we get

$$L_2x(t) - L_2x(t_0) = \int_{t_0}^{t} L_3x(s)\Delta s,$$

hence

$$p(t)\{(L_1x)^{\Delta}\}^{\alpha}(t) \geq (t - t_0)L_3x(t),$$

thus

$$x^{\Delta^2}(t) \geq \left(\frac{t-t_0}{p(t)}\right)^{\frac{1}{\alpha}} L_3^{\frac{1}{\alpha}}x(t) \quad \text{for} \quad t \geq t_0. \tag{4}$$

Integrating (4) from t_0 to t, using (I) and the decreasing property of $L_3x(t),\ t \geq t_0$, we obtain

$$x^{\Delta}(t) \geq L_3^{\frac{1}{\alpha}}x(t)\int_{t_0}^{t}\left(\frac{s-t_0}{p(s)}\right)^{\frac{1}{\alpha}}\Delta s \quad \text{for} \quad t \geq t_0,$$

and repeating the same process yields

$$x(t) \geq L_3^{\frac{1}{\alpha}}x(t)\int_{t_0}^{t}\left(\int_{t_0}^{u}\left(\frac{s-t_0}{p(s)}\right)^{\frac{1}{\alpha}}\Delta s\right)\Delta u \quad \text{for} \quad t \geq t_0.$$

Case (II). Suppose $L_0x(t) > 0$, $L_1x(t) > 0$, $L_2x(t) < 0$, $L_3x(t) > 0$, and $L_4x(t) < 0$ for $t \geq t_0$. From $\sup \mathbb{T} = \infty$, we see that there exists an increasing function $\beta : \mathbb{T} \to \mathbb{T}$ such that $t < \beta(t)$ for all $t \in \mathbb{T}$. Then note that $L_3x(t) > 0$ is decreasing and $L_2x(t) < 0$ for $t \geq t_0$. From

$$L_2x(\beta(t)) - L_2x(t) = \int_{t}^{\beta(t)} L_3x(s)\Delta s,$$

we get that

$$-L_2x(t) \geq (\beta(t) - t)L_3x(\beta(t)),$$

which can be rewritten as

$$-x^{\Delta^2}(t) \geq \left(\frac{\beta(t)-t}{p(t)}\right)^{\frac{1}{\alpha}} L_3^{\frac{1}{\alpha}}x(\beta(t)). \tag{5}$$

Integrating (5) again from t to $\beta(t)$, we obtain

$$x^{\Delta}(t) \geq L_3^{\frac{1}{\alpha}}x(\beta^2(t))\int_{t}^{\beta(t)}\left(\frac{\beta(s)-s}{p(s)}\right)^{\frac{1}{\alpha}}\Delta s,$$

and so

$$x(t) \geq L_3^{\frac{1}{\alpha}}x(\beta^2(t))\int_{t_0}^{t}\int_{s}^{\beta(s)}\left(\frac{\beta(u)-u}{p(u)}\right)^{\frac{1}{\alpha}}\Delta u\Delta s.$$

For $t \geq t_0$, we set

$$h(t,t_0;p) := \min\left\{ \int_{t_0}^{t} \left(\frac{s-t_0}{p(s)}\right)^{\frac{1}{\alpha}} \Delta s,\ \int_{t}^{\beta(t)} \left(\frac{\beta(s)-s}{p(s)}\right)^{\frac{1}{\alpha}} \Delta s \right\}$$

and

$$H(t,t_0;p) := \int_{t_0}^{t} h(s,t_0;p(s))\Delta s.$$

Combining these inequalities, we obtain the following crucial lemma.

Lemma 2.1 *Let $x(t)$ be a positive solution of* (1) *for $t \geq t_0$. Then, for all $t \geq t_0$,*

$$x^{\Delta}(t) \geq h(t,t_0;p) L_3^{\frac{1}{\alpha}} x(\tau(t))$$

and

$$x(t) \geq H(t,t_0;p) L_3^{\frac{1}{\alpha}} x(\tau(t)).$$

We also need the following lemma.

Lemma 2.2 *[14] If X and Y are nonnegative, then*

$$X^{\lambda} - \lambda X Y^{\lambda-1} + (\lambda-1)Y^{\lambda} \geq 0,\ \lambda > 1,$$

where equality holds if and only if $X = Y$.

The following chain rule is extracted from [13] and plays an important role in this paper.

Lemma 2.3 *Assume that $\tau : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \tau(\mathbb{T}) \subset \mathbb{T}$ is a time scale such that $\tau \circ \sigma = \sigma \circ \tau$. Let $x : \tilde{\mathbb{T}} \to \mathbb{R}$. If $\tau^{\Delta}(t)$ and $x^{\Delta}(\tau(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then $(x \circ \tau)^{\Delta}(t)$ exists, and $(x \circ \tau)^{\Delta}(t) = x^{\Delta}(\tau(t))\tau^{\Delta}(t)$.*

From Remarks 4.1 and 4.2 in [6], we have the following result.

Lemma 2.4 *Assume $x \in \mathrm{C}_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$.*

(i) If $x(t) > 0,\ x^{\Delta}(t) \leq 0$ on $[t_0,\infty)_{\mathbb{T}}$ and $\lambda < 1$, then

$$\int_{t}^{\infty} \frac{-x^{\Delta}(s)}{x^{\lambda}(s)} \Delta s < \infty,\ t \in [t_0,\infty)_{\mathbb{T}};$$

(ii) If $x(t) > 0,\ x^{\Delta}(t) \geq 0$ on $[t_0,\infty)_{\mathbb{T}}$ and $\lambda > 1$, then

$$\int_{t}^{\infty} \frac{x^{\Delta}(s)}{(x^{\sigma}(s))^{\lambda}} \Delta s < \infty,\ t \in [t_0,\infty)_{\mathbb{T}}.$$

3 Oscillation Criteria for (1)

Throughout this paper, we assume $\tau \circ \sigma = \sigma \circ \tau$. In what follows we assume that

$$f^{\frac{1}{\alpha}-1}(u)g(u,v) \geq k > 0 \quad \text{for} \quad u, v \neq 0, \tag{6}$$

where k is a constant and

$$g(u,v) = \int_0^1 f'(hu + (1-h)v)dh. \tag{7}$$

Theorem 3.1 *Assume that* (2), (6) *and* (7) *hold, and there exists a function* $r : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^+$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^t \left[r(s)q(\tau(s))\tau^\Delta(s) - \left(\frac{1}{k^\alpha} \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}} \frac{(r^\Delta(t))^{1+\alpha}}{(r(t)h(t,t_0;p))^\alpha} \right) \right] \Delta s = \infty,$$

where $h(t, t_0; p)$ *is defined as in Lemma 2.1. Then equation* (1) *is oscillatory.*

Proof: Let x be an eventually positive solution of (1), say $x(t) > 0$ for all $t \geq t_0$, $t_0 \in \mathbb{T}$. Then from (1), we see that $L_4x(t) < 0$ for all $t \geq t_0$ and hence $L_ix(t)$, $i = 1, 2, 3$, are eventually of one sign for all $t \geq t_0$. From the earlier argument in Section 2, $L_3x(t) > 0$ is decreasing and $L_1x(t) > 0$ for all $t \geq t_1 \geq t_0$, $t_1 \in \mathbb{T}$. By Lemma 2.1, there exists $t_2 \geq t_1$, $t_2 \in \mathbb{T}$ such that

$$x^\Delta(t) \geq h(t, t_2; p)L_3^{\frac{1}{\alpha}}x(\tau(t)) \quad \text{for} \quad t \geq t_2. \tag{8}$$

Define

$$w(t) = r(t)\frac{L_3x(\tau(t))}{f(x(t))} \quad \text{for} \quad t \geq t_2.$$

Then by Lemma 2.3 and [8, Theorem 1.90], for $t \geq t_0$, we have

$$\begin{aligned}
w^\Delta(t) &= r^\Delta(t)\frac{L_3x(\tau(\sigma(t)))}{f(x(\sigma(t)))} + r(t)\frac{[L_3x(\tau(t))]^\Delta f(x(t)) - L_3x(\tau(t))[f(x(t))]^\Delta}{f(x(t))f(x(\sigma(t)))} \\
&= r^\Delta(t)\frac{w^\sigma(t)}{r^\sigma(t)} - r(t)\frac{q(\tau(t))f(x^\sigma(\tau(t)))\tau^\Delta(t)f(x(t))}{f(x(t))f(x(\sigma(t)))} \\
&\quad -r(t)L_3x(\tau(t))\frac{x^\Delta(t)\int_0^1 f'[(1-h)x(t) + hx^\sigma(t)]dh}{f(x(t))f(x(\sigma(t)))} \\
&\leq r^\Delta(t)\frac{w^\sigma(t)}{r^\sigma(t)} - r(t)q(\tau(t))\tau^\Delta(t) - r(t)L_3x(\tau(t))x^\Delta(t)\frac{g(x^\sigma(t), x(t))}{f(x(t))f(x(\sigma(t)))} \\
&\leq r^\Delta(t)\frac{w^\sigma(t)}{r^\sigma(t)} - r(t)q(\tau(t))\tau^\Delta(t) - kr(t)L_3x(\tau(t))\frac{x^\Delta(t)}{f^{1+\frac{1}{\alpha}}(x(\sigma(t)))} \\
&\leq r^\Delta(t)\frac{w^\sigma(t)}{r^\sigma(t)} - r(t)q(\tau(t))\tau^\Delta(t) - kr(t)L_3^{1+\frac{1}{\alpha}}x(\tau(t))h(t,t_2;p)\frac{1}{f^{1+\frac{1}{\alpha}}(x(\sigma(t)))} \\
&\leq r^\Delta(t)\frac{w^\sigma(t)}{r^\sigma(t)} - r(t)q(\tau(t))\tau^\Delta(t) - kr(t)h(t,t_2;p)\frac{w^{1+\frac{1}{\alpha}}(\sigma(t))}{r^{1+\frac{1}{\alpha}}(x(\sigma(t)))}.
\end{aligned} \tag{9}$$

Set

$$X = [kr(t)h(t,t_2;p)]^{\frac{\alpha}{\alpha+1}} \frac{w^\sigma(t)}{r^\sigma(t)}, \quad \lambda = \frac{\alpha+1}{\alpha} > 1$$

and

$$Y = \left(\frac{\alpha}{\alpha+1}\right)^\alpha \left(\frac{r^\Delta(t)}{r^\sigma(t)}\right)^\alpha \left\{[kr(t)h(t,t_2;p)]^{-\frac{\alpha}{\alpha+1}} r^\sigma(t)\right\}^\alpha$$

in Lemma 2.2 to conclude that for $t \geq t_3 > t_2$, $t_3 \in \mathbb{T}$

$$r^\Delta(t)\frac{w^\sigma(t)}{r^\sigma(t)} - kr(t)h(t,t_2;p)\frac{w^{1+\frac{1}{\alpha}}(\sigma(t))}{r^{1+\frac{1}{\alpha}}(\sigma(t))} \leq \frac{1}{k^\alpha}\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\frac{(r^\Delta(t))^{1+\alpha}}{(r(t)h(t,t_2;p))^\alpha},$$

and so

$$w^\Delta(t) \leq -r(t)q(\tau(t))\tau^\Delta(t) + \frac{1}{k^\alpha}\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\frac{(r^\Delta(t))^{1+\alpha}}{(r(t)h(t,t_2;p))^\alpha}, \ t \geq t_3. \tag{10}$$

Integrating both of sides of (10) from t_3 to $t \geq t_3$, we obtain

$$w(t)-w(t_3) \leq -\int_{t_3}^t \left[r(s)q(\tau(s))\tau^\Delta(s) - \left(\frac{1}{k^\alpha}\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\frac{(r^\Delta(s))^{1+\alpha}}{(r(s)h(s,t_2;p))^\alpha}\right)\right]\Delta s \to -\infty \text{ as } t \to \infty,$$

which contradicts the fact that $w(t) > 0$ for $t \geq t_2$. □

Now let

$$Q(t) := \int_t^\infty q(\tau(s))\tau^\Delta(s)\Delta s \ \text{ and } \ Q^*(t) := r(t)Q(t).$$

For the next three results, we obtain the oscillation criteria for (1) and special cases of (1) depending on α.

Theorem 3.2 *Let $0 < \alpha \leq 1$ and assume that (2), (6) and (7) hold. If there exists a function $r:\ [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}^+$ such that $Q^*(t) > 0$ and*

$$\limsup_{t\to\infty} \int_{t_0}^t \left[p(s)q(\tau(s))\tau^\Delta(s) - \frac{1}{4k}\frac{(r^\Delta(s))^2 Q^{1-\frac{1}{\alpha}}(\sigma(s))}{r(s)h(s,t_0;p)}\right]\Delta s = \infty,$$

where $h(t,t_0;p)$ is defined as in Lemma 2.1, then equation (1) is oscillatory.

Proof: Let x be a nonoscillatory solution of equation (1), say $x(t) > 0$ for $t \geq t_0$, $t_0 \in \mathbb{T}$. Define

$$y(t) = \frac{L_3 x(\tau(t))}{f(x(t))} \ \text{ for } \ t \geq t_0.$$

Then, similar to the proof of Theorem 3.1, we have $y^\Delta(t) \leq -q(\tau(t))\tau^\Delta(t)$, and so $y(t) \geq Q(t)$ for $t \geq t_0$. Next, we define

$$w(t) = r(t)\frac{L_3 x(\tau(t))}{f(x(t))} \ \text{ for } \ t \geq t_0.$$

Then $w(t) \geq r(t)Q(t) = Q^*(t),\ t \geq t_0$. Proceeding as in the proof of Theorem 3.1, we obtain (9) for $t \geq t_2$, $t_2 \in \mathbb{T}$. Now, for $t \geq t_3$, $t_3 \in \mathbb{T}$, $t_3 \geq t_2$ we obtain

$$\begin{aligned}
w^{\Delta}(t) &\leq r^{\Delta}(t)\frac{w^{\sigma}(t)}{r^{\sigma}(t)} - r(t)q(\tau(t))\tau^{\Delta}(t) - kr(t)h(t,t_2;p)\frac{w^{1+\frac{1}{\alpha}}(\sigma(t))}{r^{1+\frac{1}{\alpha}}(x(\sigma(t)))} \\
&\leq r^{\Delta}(t)\frac{w^{\sigma}(t)}{r^{\sigma}(t)} - r(t)q(\tau(t))\tau^{\Delta}(t) - kr(t)h(t,t_2;p)r^{-1-\frac{1}{\alpha}}(\sigma(t))(Q^*)^{\frac{1}{\alpha}-1}(\sigma(t))w^2(\sigma(t) \\
&= -r(t)q(\tau(t))\tau^{\Delta}(t) + \frac{1}{4k}\frac{(r^{\Delta}(t))^2 Q^{1-\frac{1}{\alpha}}(\sigma(t))}{r(t)h(t,t_2;p)} \\
&- \left[\sqrt{kr(t)r^{-2}(\sigma(t))h(t,t_2;p)Q^{\frac{1}{\alpha}-1}(\sigma(t))}\,w^{\sigma}(t) - \frac{r^{\Delta}(t)}{2r^{\sigma}(t)\sqrt{kr(t)r^{-2}(\sigma(t))h(t,t_2;p)Q^{\frac{1}{\alpha}-1}(\sigma(t))}}\right]^2 \\
&\leq -r(t)q(\tau(t))\tau^{\Delta}(t) + \frac{1}{4k}\frac{(r^{\Delta}(t))^2 Q^{1-\frac{1}{\alpha}}(\sigma(t))}{r(t)h(t,t_2;p)}.
\end{aligned}$$

The rest of the proof is similar to that of Theorem 3.1 and hence omitted. □

The following result is concerned with the oscillation of a special case of equation (1), namely, the equation

$$L_4(x(t)) + q(t)x^{\alpha}(\sigma(t)) = 0. \tag{11}$$

Theorem 3.3 *Let $\alpha \geq 1$ and assume that* (2) *holds. If there exists a function $r:\ [t_0,\infty)_{\mathbb{T}} \to \mathbb{R}^+$ such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[p(s)q(\tau(s))\tau^{\Delta}(s) - \frac{(r^{\Delta}(t))^2}{4\alpha r(t)h^{\alpha}(t,t_0;p)(t-t_0)^{\alpha-1}}\right]\Delta s = \infty,$$

where $h(t,t_0;p)$ is defined as in Lemma 2.1, then all bounded solutions of equation (11) *are oscillatory.*

Proof: Let $x(t)$ be an eventually bounded positive solution of equation (11). It is easy to see that $x(t)$ satisfies (II). Proceeding as in the proof of Theorem 3.1, we obtain (8) for $t \geq t_2$, $t_2 \in \mathbb{T}$, and we can easily see that

$$x(t) \geq (t-t_2)x^{\Delta}(t),\ t \geq t_2. \tag{12}$$

Define

$$w(t) = r(t)\frac{L_3(x(\tau(t)))}{x^{\alpha}(t)},\ t \geq t_2.$$

By Lemma 2.3, we can show that $[x^\alpha(t)]^\Delta \geq \alpha x^\Delta(t) x^{\alpha-1}(t)$. Then from (8), for $t \geq t_2$, we have

$$\begin{aligned}
w^\Delta(t) &= r^\Delta(t)\frac{L_3x(\tau(\sigma(t)))}{x^\alpha(\sigma(t))} + r(t)\frac{x^\alpha(t)(L_3x(\tau(t)))^\Delta - L_3x(\tau(t))(x^\alpha(t))^\Delta}{x^\alpha(t)x^\alpha(\sigma(t))} \\
&\leq \frac{r^\Delta(t)}{r^\sigma(t)}w^\sigma(t) - r(t)q(\tau(t))\tau^\Delta(t) - r(t)L_3x(\tau(t))\frac{(x^\alpha(t))^\Delta}{x^\alpha(t)x^\alpha(\sigma(t))} \\
&\leq \frac{r^\Delta(t)}{r^\sigma(t)}w^\sigma(t) - r(t)q(\tau(t))\tau^\Delta(t) - \alpha r(t)L_3x(\tau(t))(x^\Delta(t))^\alpha(x^\Delta(t))^{1-\alpha}\frac{x^{\alpha-1}(t)}{(x^\alpha(\sigma(t)))^2} \\
&\leq \frac{r^\Delta(t)}{r^\sigma(t)}w^\sigma(t) - r(t)q(\tau(t))\tau^\Delta(t) - \alpha\frac{r(t)}{r^2(\sigma(t))}h^\alpha(t,t_2;p)\left(\frac{x(t)}{x^\Delta(t)}\right)^{\alpha-1}w^2(\sigma(t)).
\end{aligned}$$

Using (12) in the above inequality, we get for $t \geq t_3 > t_2$, $t_3 \in \mathbb{T}$,

$$\begin{aligned}
w^\Delta(t) &\leq \frac{r^\Delta(t)}{r^\sigma(t)}w^\sigma(t) - r(t)q(\tau(t))\tau^\Delta(t) - \alpha\frac{r(t)}{r^2(\sigma(t))}(t-t_2)^{\alpha-1}h^\alpha(t,t_2;p)w^2(\sigma(t)) \\
&= -r(t)q(\tau(t))\tau^\Delta(t) + \frac{(r^\Delta(t))^2}{4\alpha r(t)h^\alpha(t,t_0;p)(t-t_0)^{\alpha-1}} \\
&\quad - \left[\sqrt{\alpha\frac{r(t)}{r^2(\sigma(t))}h^\alpha(t,t_2;p)(t-t_2)^{\alpha-1}}w^\sigma(t) - \frac{r^\Delta(t)}{2r^\sigma(t)\sqrt{\alpha\frac{r(t)}{r^2(\sigma(t))}h^\alpha(t,t_2;p)(t-t_2)^{\alpha-1}}}\right]^2 \\
&\leq -\left[r(t)q(\tau(t))\tau^\Delta(t) - \frac{(r^\Delta(t))^2}{4\alpha r(t)h^\alpha(t,t_0;p)(t-t_0)^{\alpha-1}}\right].
\end{aligned}$$

The rest of the proof is similar to that of Theorem 3.1, and hence omitted. □

The following result is concerned with the oscillation of another special case of equation (1), namely

$$L_4x(t) + q(t)x^\sigma(t) = 0 \tag{13}$$

Theorem 3.4 *Let $\alpha < 1$. In addition to (2) we assume that*

$$\int^\infty q(\tau(t))\tau^\Delta(t)\Delta t < \infty.$$

and

$$\lim_{t\to\infty}\int_{t_0}^t h(s,t_0;p)Q^{\frac{1}{\alpha}}(\sigma(s))\Delta s = \infty, \quad t_0 \in \mathbb{T}. \tag{14}$$

Then equation (13) is oscillatory.

Proof: Let $x(t)$ be an eventually positive solution of equation (13), say, $x(t) > 0$ for $t \geq t_0$. We define

$$w(t) = \frac{L_3x(\tau(t))}{x(t)}.$$

Similar to the proof in Theorem 3.1, we obtain

$$\left(\frac{L_3x(\tau(t))}{x(t)}\right)^{\Delta} \le -q(\tau(t))\tau^{\Delta}(t),\ t \ge t_2. \tag{15}$$

Integrating (15) from $\sigma(t)$ to u, we get

$$0 < \frac{L_3x(\tau(u))}{x(u)} \le \frac{L_3x(\tau(\sigma(t)))}{x(\sigma(t))} - \int_{\sigma(t)}^{u} q(\tau(s))\tau^{\Delta}(s)\Delta s. \tag{16}$$

Letting $u \to \infty$ in (16), we obtain

$$\frac{L_3x(\tau(\sigma(t)))}{x(\sigma(t))} \ge Q(\sigma(t)). \tag{17}$$

Using (8) in (17) and noting that L_3x is decreasing, we find

$$\left(\frac{x^{\Delta}(t)}{x^{\frac{1}{\alpha}}(\sigma(t))}\right)^{\alpha} \ge h^{\alpha}(t, t_2; p)Q(\sigma(t)),\ t \ge t_2$$

or

$$h(t, t_2; p)Q^{\frac{1}{\alpha}}(\sigma(t)) \le \frac{x^{\Delta}(t)}{x^{\frac{1}{\alpha}}(\sigma(t))},\ t \ge t_2 \tag{18}$$

Integrating (18) from t_2 to t, we get

$$\int_{t_2}^{t} h(s, t_2; p)Q^{\frac{1}{\alpha}}(\sigma(s))\Delta s \le \int_{t_2}^{t} \frac{x^{\Delta}(s)}{x^{\frac{1}{\alpha}}(\sigma(s))}\Delta s. \tag{19}$$

Taking limit of both sides of (19) as $t \to \infty$, we arrive at the desired contradiction by Lemma 2.4 (ii) and (14). $\square$

4 Oscillation Criteria for Delay Dynamic Equations

We consider the delay dynamic equation

$$L_4x(t) + q(t)f(x(\theta(t))) = 0, \tag{20}$$

where $\theta : \mathbb{T} \to \mathbb{T}$ is an increasing delay function satisfying $\lim_{t\to\infty} \theta(t) = \infty$ and $\theta(t) \le t$ for all $t \in \mathbb{T}$ and we study the oscillation for delay dynamic equation (20). We first present some comparison criteria.

Lemma 4.1 *Let* (2) *hold. If the inequality*

$$L_4x(t) + q(t)f(x(\theta(t))) \le 0 \tag{21}$$

has an eventually positive (negative) solution, then the equation (20) *also has an eventually positive (negative) solution.*

Proof: Let $x(t)$ be an eventually positive solution of (21). There exists $t_0 \in \mathbb{T}$ such that $x(t) > 0$ for $t \geq t_0$ and $x(t)$ satisfies either (I) or (II) for $t \geq t_0$. Integrating (21) from $t \geq t_0$ to $u \geq t$ and letting $u \to \infty$, we have

$$L_3 x(t) \geq \int_t^\infty q(s) f(x(\theta(s)))\Delta s. \tag{22}$$

Now, we need to distinguish the following two cases:

Case (I). $L_i x(t) > 0$ for $t \geq t_0$, $i = 0, 1, 2, 3$. Integrating (22) from t_0 to $t \geq t_0$, we get

$$L_2 x(t) \geq \int_{t_0}^t \int_s^\infty q(u) f(x(\theta(u)))\Delta u \Delta s$$

or

$$x^{\Delta^2}(t) \geq \left(\frac{1}{p(t)} \int_{t_0}^t \int_s^\infty q(u) f(x(\theta(u)))\Delta u \Delta s \right)^{\frac{1}{\alpha}}$$

and so,

$$x(t) \geq \int_{t_0}^t \int_{t_0}^{s_3} \left(\frac{1}{p(s_2)} \int_{t_0}^{s_2} \int_{s_1}^\infty q(s) f(x(\theta(s)))\Delta s \Delta s_1 \right)^{\frac{1}{\alpha}} \Delta s_2 \Delta s_3.$$

Define

$$\Phi(t, x(\theta(t))) := \int_{t_0}^t \int_{t_0}^{s_3} \left(\frac{1}{p(s_2)} \int_{t_0}^{s_2} \int_{s_1}^\infty q(s) f(x(\theta(s)))\Delta s \Delta s_1 \right)^{\frac{1}{\alpha}} \Delta s_2 \Delta s_3 - c,$$

where $c = x(t_0)$. Now, we show that the existence of a positive solution to the equation

$$y(t) = c + \Phi(t, y(\theta(t))) \text{ for } t \geq t_0.$$

In order to do this, we define the function sequence $\{y_k(t)\}$, $k = 0, 1, \ldots$, such that

$$y_0(t) = x(t), \quad y_{k+1}(t) = c + \Phi(t, y_k(\theta(t))) \text{ for } t \geq t_0. \tag{23}$$

Then, one can easily see that $y_k(t)$ is well-defined and

$$0 \leq y_k(t) \leq x(t), \quad c \leq y_{k+1}(t) \leq y_k(t).$$

Thus, y_k is positive and nonincreasing in k for $t \geq t_0$. This means we may define $y(t) = \lim\limits_{k\to\infty} y_k(t)$. Since $0 < y(t) \leq y_k(t) \leq x(t)$ for all $k \geq 0$ and

$$\Phi(t, y_k(\theta(t))) \leq \Phi(t, x(\theta(t))),$$

the convergence of $\{y_k(t)\}$ is uniform with respect to t. Now, taking the limit of both sides of (23), we have

$$y(t) = c + \Phi(t, y(\theta(t))) \;\text{ for }\; t \geq t_0. \tag{24}$$

Finally, taking the delta derivative of (24) four times, we obtain

$$L_4 y(t) + q(t) f(y(\theta(t))) = 0.$$

Case (II). $L_i x(t) > 0,\ i = 0, 1, 3$ and $L_2 x(t) < 0$ for $t \geq t_0$. Integrating (22) from $t \geq t_0$ to u and letting $u \to \infty$, we have

$$-x^{\Delta^2}(t) \geq \left(\frac{1}{p(t)} \int_t^{\infty} \int_s^{\infty} q(u) f(x(\theta(u))) \Delta u \Delta s \right)^{\frac{1}{\alpha}}$$

Define

$$\Psi(t, x(\theta(t))) := \int_{t_0}^{t} \int_{s_3}^{\infty} \left(\frac{1}{p(s_2)} \int_{s_2}^{\infty} \int_{s_1}^{\infty} q(s) f(x(\theta(s))) \Delta s \Delta s_1 \right)^{\frac{1}{\alpha}} \Delta s_2 \Delta s_3.$$

Then

$$x(t) \geq c + \Psi(t, x(\theta(t))) \text{ with } c = x(t_0).$$

The rest of the proof is similar to that of Case (I) and hence omitted. □

Theorem 4.2 *Let* (2) *hold. If*

$$x^{\Delta^2}(t) + \left(\frac{1}{p(t)} \int_t^{\infty} \int_s^{\infty} q(u) \Delta u \right)^{\frac{1}{\alpha}} f^{\frac{1}{\alpha}}(x(\sigma(t)) = 0 \tag{25}$$

is oscillatory, then all bounded solutions of equation (1) *are oscillatory.*

Proof: Let $x(t)$ be a nonoscillatory bounded solution of equation (1), say, $x(t) > 0$ for $t \geq t_0$, $t_0 \in \mathbb{T}$. It is easy to check that $x(t)$ satisfies (II) for $t \geq t_1 \geq t_0$, $t_1 \in \mathbb{T}$. Integrating equation (1) from $t \geq t_1$ to u and letting $u \to \infty$, we have

$$L_3 x(u) - L_3 x(t) = -\int_t^u q(s) f(x^{\sigma}(s)) \Delta s$$

and so

$$L_3 x(t) \geq \int_t^{\infty} q(s) f(x^{\sigma}(s)) \Delta s \geq f(x^{\sigma}(t)) \int_t^{\infty} q(s) \Delta s,\ t \geq t_1. \tag{26}$$

Once again, integrating (26) from $t \geq t_1$ to u and letting $u \to \infty$, we obtain

$$-L_2 x(t) \geq f(x^{\sigma}(t)) \int_t^{\infty} \left(\int_s^{\infty} q(v) \Delta v \right) \Delta s,\ t \geq t_1$$

or

$$x^{\Delta^2}(t) + \left(\frac{1}{p(t)} \int_t^{\infty} (\int_s^{\infty} q(v) \Delta v) \Delta s \right)^{\frac{1}{\alpha}} f^{\frac{1}{\alpha}}(x^{\sigma}(t)) \leq 0,\ t \geq t_1.$$

Similar to the proof of Case (II) in Lemma 4.1, we can show that (25) has an eventually positive solution, which contradicts the hypothesis and completes the proof. □

By the similar method of proof in Lemma 4.1, we can show that the following comparison result.

Lemma 4.3 *Let* (2) *hold. If*

$$x^{\Delta}(t) + q(t)f(x(\theta(t)) \leq 0,$$

has an eventually positive (negative) solution, then

$$x^{\Delta}(t) + q(t)f(x(\theta(t)) = 0$$

also has an eventually positive (negative) solution.

Now we establish some oscillation criteria for equation (20).

Theorem 4.4 *Let* (2) *hold and assume that* f *satisfies*

$$-f(-xy) \geq f(xy) \geq f(x)f(y) \quad \textit{for} \quad xy > 0. \tag{27}$$

If

$$y^{\Delta}(t) + q(t)f(H(\theta(t), t_0; p))f(y^{\frac{1}{\alpha}}(\tau(\theta(t)))) = 0 \tag{28}$$

is oscillatory, then equation (20) *is oscillatory.*

Proof: Let $x(t)$ be a nonoscillatory solution of equation (20), say, $x(t) > 0$ for $t \geq t_0$, $t_0 \in \mathbb{T}$, and $x(t)$ satisfies (I) or (II). By Lemma 2.1, there exists $t_1 \geq t_0$ so large that

$$x(\theta(t)) \geq H(\theta(t), t_0; p)L_3^{\frac{1}{\alpha}}x(\tau(\theta(t))) \text{ for } t \geq t_1, t_1 \in \mathbb{T}. \tag{29}$$

Using (29) and (27) in (20), we have

$$-(L_3x(t))^{\Delta} \geq q(t)f(H(\theta(t)), t_0; p))f(L_3^{\frac{1}{\alpha}}x(\tau(\theta(t)))), \; t \geq t_1.$$

Substituting $y(t)$ for $L_3x(t)$, $t \geq t_1$, we get

$$-y^{\Delta}(t) \geq q(t)f(H(\theta(t)), t_0; p))f(y^{\frac{1}{\alpha}}(\tau(\theta(t)))), \; t \geq t_1.$$

Note that $x(t)$ satisfies (I) or (II) implies that $y(t) > 0$ for $t \geq t_1$. So (28) has an eventually positive solution by Lemma 4.3, which contradicts with the hypothesis and completes the proof. □

We now present the following result for the oscillation of all bounded solutions of equation (20).

Theorem 4.5 *Let* (2) *hold. If*

$$\int_{t_0}^{\infty}\int_{k}^{\infty}\left(\frac{1}{p(\nu)}\int_{\nu}^{\infty}\int_{\tau}^{\infty}q(s)\Delta s\Delta\tau\right)^{\frac{1}{\alpha}}\Delta\nu\Delta k = \infty, \tag{30}$$

then all bounded solutions of equation (20) *are oscillatory.*

Proof: Let $x(t)$ be a nonoscillatory bounded solutions of equation (20), say, $x(t) > 0$ for $t \ge t_0$. Clearly, $x(t)$ satisfies (II) for $t \ge t_1$ for some $t_1 \ge t_0$. Now, there exist a constant $c > 0$ and an $t_2 \ge t_1$, $t_2 \in \mathbb{T}$ such that

$$\frac{c}{2} \le x(\theta(t)) \le c \ \text{ for } \ t \ge t_2. \tag{31}$$

Using (31), we have

$$L_4 x(t) + q(t) f(\frac{c}{2}) \le 0 \ \text{ for } \ t \ge t_2. \tag{32}$$

Integrating (32) from $t \ge t_2$ to $u \ge t$ and letting $u \to \infty$, we have

$$L_3 x(t) \ge f(\frac{c}{2}) \int_t^\infty q(s)\Delta s. \tag{33}$$

Once again, integrating (33) from $t \ge t_2$ to $u \ge t$ and letting $u \to \infty$, we get

$$-x^{\Delta^2}(t) \ge f^{\frac{1}{\alpha}}(\frac{c}{2}) \left(\frac{1}{p(t)} \int_t^\infty \int_\tau^\infty q(s)\Delta s \Delta\tau \right)^{\frac{1}{\alpha}}.$$

Therefore, we find

$$x(t) \ge x(t_1) + f^{\frac{1}{\alpha}}(\frac{c}{2}) \int_{t_2}^\infty \int_k^\infty \left(\frac{1}{p(\nu)} \int_\nu^\infty \int_\tau^\infty q(s)\Delta s \Delta\tau \right)^{\frac{1}{\alpha}} \Delta\nu\Delta k \to \infty \text{ as } t \to \infty,$$

which contradicts with (30) and completes the proof. □

The following theorems are concerned with a necessary and sufficient condition for the oscillation of all bounded and unbounded solutions of

$$L_4 x(t) + q(t) x^\gamma(\theta(t)) = 0, \tag{34}$$

where γ is the ratio of two positive odd integers.

Theorem 4.6 *Let $\gamma > \alpha$ and condition* (2) *hold. Then all bounded solutions of equation* (34) *are oscillatory if and only if* (30) *holds.*

Proof: Let $x(t)$ be a nonoscillatory bounded solution of equation (34), say $x(t) > 0$ for $t \ge t_0$, $t_0 \in \mathbb{T}$. Clearly, $x(t)$ satisfies (II) for $t \ge t_1$ for some $t_1 \ge t_0$, $t_1 \in \mathbb{T}$. The proof of the "if" part is an immediate corollary of Theorem 4.5 and hence omitted.

Now, we prove the "only if" part of the theorem. Let $c > 0$ be a given arbitrary constant, and choose a large $T \ge t_1$, $T \in \mathbb{T}$ such that

$$\int_T^\infty \int_k^\infty \left(\frac{1}{p(\nu)} \int_\nu^\infty \int_\tau^\infty q(s)\Delta s \Delta\tau \right)^{\frac{1}{\alpha}} \Delta\nu\Delta k \le \frac{1}{2} c^{1-\frac{\gamma}{\alpha}}.$$

We introduce the Banach space $\mathcal{A}$ of all bounded real-valued functions defined on $[T,\infty)_\mathbb{T}$ with the norm $||x|| = \sup_{t \in [T,\infty)_\mathbb{T}} |x(t)|$. We define a bounded convex and closed subset $\mathcal{B}$ of $\mathcal{A}$ as

$$\mathcal{B} = \{x \in \mathcal{A} : \ \frac{c}{2} \le x(\theta(t)) \le c, \ t \ge T\}.$$

Next, let $\mathcal{F}$ be a mapping defined on $\mathcal{B}$ by

$$(\mathcal{F}x)(t) = c - \int_T^t \int_k^\infty \left(\frac{1}{p(\nu)} \int_\nu^\infty \int_\tau^\infty q(s) x^\gamma(\theta(s)) \Delta s \Delta \tau \right)^{\frac{1}{\alpha}} \Delta \nu \Delta k.$$

It is now easy to show that $\mathcal{F}$ maps $\mathcal{B}$ into itself and $\mathcal{F}$ is a continuous mapping. Also $\mathcal{F}(\mathcal{B})$ is relatively compact in $\mathcal{A}$. Therefore, by Schauder fixed point theorem, there exists $x \in \mathcal{B}$ such that $x = \mathcal{F}x$. It is clear that the fixed point $x = x(t)$ gives a positive solution of (34) for $t \geq T$. □

Theorem 4.7 *Let $\gamma < \alpha$ and* (2) *hold. Then all unbounded solutions of* (34) *are oscillatory if and only if for all large $t \geq t_0$, $t_0 \in \mathbb{T}$,*

$$\int^\infty q(t) H_1^\gamma(\theta(t), t_0; p) \Delta t = \infty, \tag{35}$$

where

$$H_1(t, t_0; p) = \int_{t_0}^t \int_{t_0}^u \left(\frac{s - t_0}{p(s)} \right)^{\frac{1}{\alpha}} \Delta s \Delta u.$$

Proof: Let $x(t)$ be an unbounded nonoscillatory solution of equation (34), say $x(t) > 0$ for $t \geq t_0$, $t_0 \in \mathbb{T}$. It is easy to see that $x(t)$ satisfies (I), and so,

$$x(t) \geq H_1(t, t_0; p) L_3^{\frac{1}{\alpha}} x(t), \ t \geq t_1 \geq t_0, \ t_1 \in \mathbb{T}$$

Furthermore, we have

$$x^\gamma(\theta(t)) \geq H_1^\gamma(\theta(t), t_0; p) L_3^{\frac{\gamma}{\alpha}} x(\theta(t)) \geq H_1^\gamma(\theta(t), t_0; p) L_3^{\frac{\gamma}{\alpha}} x(t). \tag{36}$$

Using (36) in (34), we obtain

$$-L_4 x(t) \geq q(t) H_1^\gamma(\theta(t), t_0; p) L_3^{\frac{\gamma}{\alpha}} x(t).$$

Set $y(t) = L_3 x(t)$. Then

$$y^\Delta(t) + q(t) H_1^\gamma(\theta(t), t_0; p) y^{\frac{\gamma}{\alpha}}(t) \leq 0. \tag{37}$$

Dividing both sides of (37) by $y^{\frac{\gamma}{\alpha}}(t)$ and integrating from t_1 to u

$$\int_{t_1}^u q(t) H_1^\gamma(\theta(t), t_0; p) \Delta t \leq - \int_{t_1}^u \frac{y^\Delta(t)}{y^{\frac{\gamma}{\alpha}}(t)} \Delta t.$$

By Lemma 2.4 (i) and as $u \to \infty$, we conclude that

$$\int_{t_1}^\infty q(t) H_1^\gamma(\theta(t), t_0; p) \Delta t \leq \int_{t_1}^\infty \frac{-y^\Delta(t)}{y^{\frac{\gamma}{\alpha}}(t)} \Delta t < \infty,$$

which contradicts with (35).

To prove the "only if" part it suffices to assume that

$$\int^{\infty} q(s)H_1^{\gamma}(\theta(s), t_0; p)\Delta s < \infty,$$

and show the existence of a nonoscillatory solution of (34). Let $c > 0$ be an arbitrary constant and choose $T > t_1 \geq t_0$, $T \in \mathbb{T}$ sufficiently large so that

$$\int_T^{\infty} q(s)H_1^{\gamma}(\theta(s), t_0; p)\Delta s < c^{1-\frac{\alpha}{\gamma}}.$$

Let X be the subset of all real-valued functions set X_1 defined on $[T, \infty)_{\mathbb{T}}$ by

$$X = \{x : \ c_1 H_1(t, T; p) \leq x(t) \leq c_2 H_1(t, T; p), \ t \geq T\},$$

where $c_1 = (\frac{c}{2})^{\frac{1}{\alpha}}$ and $c_2 = (2c)^{\frac{1}{\alpha}}$. Clearly, X is a closed, convex and compact subset of X_1. Let S be a mapping defined on X as follows: For $x \in X$,

$$(Sx)(t) = \int_T^t \int_T^k \left(\frac{1}{p(\nu)} \left[c(\nu - T) + \int_T^{\nu} \int_{\tau}^{\infty} q(s)x^{\gamma}(\theta(s))\Delta s \Delta\tau \right] \right)^{\frac{1}{\alpha}} \Delta\nu\Delta k \ \text{ for } \ t \geq T. \tag{38}$$

It is easy to show that S is continuous and maps X into itself and relatively compact in X_1. Therefore, by Schauder fixed point theorem, S has a fixed point x in X which satisfies

$$x(t) = \int_T^t \int_T^k \left(\frac{1}{p(\nu)} \left[c(\nu - T) + \int_T^{\nu} \int_{\tau}^{\infty} q(s)x^{\gamma}(\theta(s))\Delta s \Delta\tau \right] \right)^{\frac{1}{\alpha}} \Delta\nu\Delta k \ \text{ for } \ t \geq T. \tag{39}$$

Taking the delta derivative four times on (39), we see that $x = x(t)$ is a positive solution of (34) for $t \geq T$. □

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