ON THE OSCILLATION OF HIGHER ORDER NEUTRAL DIFFERENCE EQUATIONS OF MIXED TYPE

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ABSTRACT. In this paper we shall establish some new criteria for the oscillation of neutral difference equations of the form

$$\Delta^m\big(x(k)+ax[k-\tau]+bx[k+\sigma]\big)+qx[k-g]+px[k+h]=0,$$

where $m \geq 1$ is an integer, a, b are real constants and p, q are nonnegative real constants, τ, σ, g, h are nonnegative integers.

By a new technique based on the characteristic equation associated to the equation under consideration, we give weaker conditions than those obtained by the authors.

1. Introduction

In this paper we shall consider the higher order neutral difference equations

$$\Delta^m\left(x(k)+ax[k-\tau]-bx[k+\sigma]\right)+qx[k-g]+px[k+h]=0, \tag{1.1}$$

$$\Delta^m\left(x(k)-ax[k-\tau]+bx[k+\sigma]\right)+qx[k-g]+px[k+h]=0, \tag{1.2}$$

$$\Delta^m\left(x(k)+ax[k-\tau]+bx[k+\sigma]\right)+qx[k-g]+px[k+h]=0 \tag{1.3}$$

and

$$\Delta^m\left(x(k)-ax[k-\tau]-bx[k+\sigma]\right)+qx[k-g]+px[k+h]=0, \tag{1.4}$$

where $m \geq 1$ is an integer, τ, σ, g, h are nonnegative integers, and a, b, p, q are nonnegative real constants.

Let Δ be the first order forward difference operator $\Delta x(k)=x(k+1)-x(k)$ and for $i \geq 1$, Δ^i be the ith order forward difference operator $\Delta^i x(k)=\Delta(\Delta^{i-1}x(k))$. A solution $\{x(k)\}$ of equation (1.j), $j=1$, 2, 3, or 4 is said to *oscillate* if for every $n_0 \geq 0$ there exists $n \geq n_0$ such that

$$x(n)x(n+1) \leq 0.$$

Otherwise the solution is called *nonoscillatory*. Equation (1.j), $j=1$, 2, 3, or 4 is called *oscillatory* if all its solutions are oscillatory.

Recently, there has been a lot of interest in the oscillations of difference equations. For recent contribution, we refer to the papers [2, 6, 8, 7]. For the general theory of difference equations the reader is referred to the monographs [1, 3, 4, 5, 9, 10].

The purpose of this paper is to establish some new easily verifiable sufficient conditions, involving the coefficients and the arguments only under which all solutions of equation (1.j), $j=1$, 2,

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3, or 4 are oscillatory. The technique employed here is based on the study of the characteristic equations

$$F_1(\lambda) := (\lambda - 1)^m \left[1 + a\lambda^{-\tau} - b\lambda^{\sigma}\right] + q\lambda^{-g} + p\lambda^h = 0, \tag{1.5}$$

$$F_2(\lambda) := (\lambda - 1)^m \left[1 - a\lambda^{-\tau} + b\lambda^{\sigma}\right] + q\lambda^{-g} + p\lambda^h = 0, \tag{1.6}$$

$$F_3(\lambda) := (\lambda - 1)^m \left[1 + a\lambda^{-\tau} + b\lambda^{\sigma}\right] + q\lambda^{-g} + p\lambda^h = 0 \tag{1.7}$$

and

$$F_4(\lambda) := (\lambda - 1)^m \left[1 - a\lambda^{-\tau} - b\lambda^{\sigma}\right] + q\lambda^{-g} + p\lambda^h = 0 \tag{1.8}$$

associated to equations (1.j), $j = 1$, 2, 3, 4 respectively.

2. Main Results

The following lemma, which will be employed in the proofs of our oscillation result is extracted from [9, 10].

Lemma 2.1. *Consider the linear difference equation*

$$x(k+m) + \sum_{j=1}^{m} q(j)x[k+m-j] = 0, \tag{2.1}$$

for $k = 0, 1, \ldots$ *where* m *is a nonnegative integer and* $q(j) \in \mathbb{R}$, $j = 1, 2, \ldots, m$. *Then the following statements are equivalent:*

- (I_1) *Every solution of* (2.1) *oscillates.*
- (I_2) *The characteristic equation associated with* (2.1)

$$\lambda^m + \sum_{j=1}^{m} q(j)\lambda^{m-j} = 0$$

has no positive roots.

First, we study the oscillatory behavior of equations (1.1) and (1.2), where g, h, τ, σ are nonnegative integers, a, b are nonnegative real constants and p, q are positive real constants.

Theorem 2.1. *Assume that* $b > 0$, $h > \sigma + m$ *and* $g > \tau$. *Moreover, suppose that*

$$p\frac{(h-\sigma)^{h-\sigma}}{m^m(h-\sigma-m)^{h-\sigma-m}} > b, \tag{2.2}$$

$$q\frac{(g-\tau+m)^{g-\tau+m}}{m^m(g-\tau)^{g-\tau}} > 1 + a \text{ if } m \text{ is odd} \tag{2.3}$$

and

$$q\frac{(g+\sigma+m)^{g+\sigma+m}}{m^m(g+\sigma)^{g+\sigma}} > b - a - 1 \text{ if } m \text{ is even.} \tag{2.4}$$

Then (1.1) *is oscillatory.*

Proof. Our strategy is to prove that under the hypotheses given above the characteristic equation (1.5) of (1.1) has no positive roots. There are three possible cases to consider:

Case 1: m is even or odd and $\lambda > 1$.

For $\lambda \neq 1$, we have

$$\begin{aligned} \frac{F_1(\lambda)}{(\lambda-1)^m}\lambda^{-\sigma} &= \frac{q\lambda^{-(g+\sigma)}+p\lambda^{h-\sigma}}{(\lambda-1)^m}+\lambda^{-\sigma}+a\lambda^{-(\tau+\sigma)}-b \\ &\geq p\frac{\lambda^{h-\sigma}}{(\lambda-1)^m}-b. \end{aligned}$$

Since the minimum of $f_1(x)=\frac{x^\alpha}{(x-1)^\beta}$, $\alpha>\beta$ and $x>1$ occurs at $x=\frac{\alpha}{\alpha-\beta}$, we see that

$$\frac{F_1(\lambda)}{(\lambda-1)^m}\lambda^{-\sigma} \geq p\frac{(\frac{h-\sigma}{h-\sigma-m})^{h-\sigma}}{(\frac{m}{h-\sigma-m})^m}-b>0.$$

Case 2: m is odd and $0<\lambda<1$.

In this case we have

$$\begin{aligned} -\frac{F_1(\lambda)}{(\lambda-1)^m}\lambda^{\tau} &= -\left(\frac{q\lambda^{-(g-\tau)}+p\lambda^{\tau+\sigma}}{(\lambda-1)^m}\right)-(\lambda^\tau+a-b\lambda^{\tau+\sigma}) \\ &= \frac{q\lambda^{-(g-\tau)}+p\lambda^{\tau+\sigma}}{(1-\lambda)^m}-(\lambda^\tau+a-b\lambda^{\tau+\sigma}) \\ &\geq q\frac{\lambda^{-(g-\tau)}}{(1-\lambda)^m}-\lambda^\tau-a \\ &\geq q\frac{\lambda^{\tau-g}}{(1-\lambda)^m}-1-a \\ &\geq q\frac{(\frac{g-\tau}{g-\tau+m})^{-(g-\tau)}}{(\frac{m}{g-\tau+m})^m}-1-a>0, \end{aligned}$$

since the minimum of the function $f_2(x)=\frac{x^{-\alpha}}{(1-x)^\beta}$ occurs at $x=\frac{\alpha}{\alpha+\beta}$ where α and β are positive.

Case 3: m is even and $0<\lambda<1$.

It follows from (2.5) that

$$\frac{F_1(\lambda)}{(\lambda-1)^m}\lambda^{-\sigma} \geq q\frac{\lambda^{-(g+\sigma)}}{(1-\lambda)^m}+\lambda^{-\sigma}+a\lambda^{-(\tau+\sigma)}-b.$$

As in Case 2, we see that

$$\frac{F_1(\lambda)}{(\lambda-1)^m}\lambda^{-\sigma} \geq q\frac{(\frac{g+\sigma}{g+\sigma+m})^{-(g+\sigma)}}{(\frac{m}{g+\sigma+m})^m}+1+a-b>0.$$

Cases 1–3 and $F_1(\lambda)>0$ on $(0,1)\cup(1,\infty)$ and $F_1(1)>0$ imply $F_1(\lambda)>0$ for $\lambda\in\mathbb{R}^+=(0,\infty)$, that is, (1.5) has no positive roots. By Lemma 2.1 we conclude that (1.1) is oscillatory. This completes the proof. □

Theorem 2.2. *Assume that $a>0$, $\tau+h>m$ and $g>\tau$. Moreover, suppose that*

$$p\frac{(h+\tau)^{h+\tau}}{m^m(h+\tau-m)^{h+\tau-m}}>a, \tag{2.5}$$

$$q\frac{(g+m)^{g+m}}{m^m g^g} > 1+b \text{ if } m \text{ is odd} \tag{2.6}$$

and

$$q\frac{(g-\tau+m)^{g-\tau+m}}{m^m (g-\tau)^{g-\tau}} > a \text{ if } m \text{ is even.} \tag{2.7}$$

Then (1.2) *is oscillatory.*

Proof. For $\lambda \neq 1$ we have

$$\frac{F_2(\lambda)\lambda^\tau}{(\lambda-1)^m} = (\lambda^\tau - a + b\lambda^{\tau+\sigma}) + \frac{q\lambda^{\tau-g} + p\lambda^{\tau+h}}{(\lambda-1)^m}. \tag{2.8}$$

Now we consider the following three cases:

Case 1: m is even or odd and $\lambda > 1$.

From (2.8), we find

$$\frac{F_2(\lambda)\lambda^\tau}{(\lambda-1)^m} \geq p\frac{\lambda^{\tau+h}}{(\lambda-1)^m} - a,$$

and since the function $f_1(x) = \frac{x^\alpha}{(x-1)^\beta}$, $x > 1$ and $\alpha > \beta$ has its minimum value at $x = \frac{\alpha}{\alpha-\beta}$, we see that

$$\frac{F_2(\lambda)\lambda^\tau}{(\lambda-1)^m} \geq p\frac{(\frac{\tau+h}{\tau+h-m})^{\tau+h}}{(\frac{m}{\tau+h-m})^m} - a > 0.$$

Case 2: m is odd and $0 < \lambda < 1$.

In this case we have

$$-\frac{F_2(\lambda)}{(\lambda-1)^m} = \frac{F_2(\lambda)}{(1-\lambda)^m} = \frac{q\lambda^{-g} + p\lambda^h}{(1-\lambda)^m} - (1 - a\lambda^{-\tau} + b\lambda^\sigma)$$

and hence

$$\frac{F_2(\lambda)}{(1-\lambda)^m} \geq \frac{q\lambda^{-g}}{(1-\lambda)^m} - 1 - b\lambda^\sigma.$$

Since the minimum of the function $f_2(x) = \frac{x^\alpha}{(1-x)^\beta}$, $0 < x < 1$ occurs at $x = \frac{\alpha}{\alpha+\beta}$, we conclude that

$$\frac{F_2(\lambda)}{(1-\lambda)^m} \geq q\frac{(\frac{g}{g+m})^{-g}}{(\frac{m}{g+m})^m} - 1 - b > 0$$

Case 3: m is even and $0 < \lambda < 1$.

From (2.8), we have

$$\begin{aligned}\frac{F_2(\lambda)\lambda^\tau}{(\lambda-1)^m} &= \frac{F_2(\lambda)\lambda^\tau}{(1-\lambda)^m}\\ &\geq q\frac{\lambda^{-(g-\tau)}}{(1-\lambda)^m} + \lambda^\tau - a + b\lambda^{\tau+\sigma}\\ &\geq q\frac{\lambda^{-(g-\tau)}}{(1-\lambda)^m} - a.\end{aligned}$$

As in Case 2, we see that

$$\frac{F_2(\lambda)\lambda^\tau}{(1-\lambda)^m} \geq q\frac{(\frac{g-\tau}{g-\tau+m})^{-(g-\tau)}}{(\frac{m}{g-\tau+m})^m} - a > 0.$$

From Cases 1–3 and $F_2(1) > 0$ we can conclude that $F_2(\lambda) > 0$ for $\lambda \in \mathbb{R}^+$, that is, (1.6) has no positive roots. Thus, the conclusion of the theorem follows by applying Lemma 2.1. ☐

Corollary 2.1. *Let m be odd, $0 < a \leq 1$ and condition (2.6) hold. Then (1.2) is oscillatory.*

Proof. Assume $\lambda \geq 1$. Since m is odd and $0 < a \leq 1$, it follows that $F_2(\lambda) > 0$. Assume $0 < \lambda < 1$, then as in the proof of Theorem 2.2 we see that $F_2(\lambda) > 0$. By applying Lemma 2.1 we can complete the proof. ☐

Next, we consider the mixed equations which are of the same form of (1.1) and (1.2)

$$\Delta^m\left(x(k) + ax[k-\tau] - bx[k-\sigma]\right) + qx[k-g] + px[k+h] = 0, \tag{2.9}$$

and

$$\Delta^m\left(x(k) - ax[k+\tau] + bx[k+\sigma]\right) + qx[k-g] + px[k+h] = 0, \tag{2.10}$$

where a, b are nonnegative real constants, g, h, τ, and σ are nonnegative integers and p, q are positive real constants.

The characteristic equations of (2.9) and (2.10) are respectively:

$$F_5(\lambda) := (\lambda-1)^m\left[1 + a\lambda^{-\tau} - b\lambda^{-\sigma}\right] + q\lambda^{-g} + p\lambda^h = 0 \tag{2.11}$$

and

$$F_6(\lambda) := (\lambda-1)^m\left[1 - a\lambda^{\tau} + b\lambda^{\sigma}\right] + q\lambda^{-g} + p\lambda^h = 0. \tag{2.12}$$

Now, we present the following oscillation criterion for (2.9).

Theorem 2.3. *Assume that $b - a - 1 > 0$, $h + \sigma > m$ and $g > \sigma \geq \tau$. Moreover, suppose that*

$$p\frac{(h+\sigma)^{h+\sigma}}{m^m(h+\sigma-m)^{h+\sigma-m}} > b - a - 1, \tag{2.13}$$

$$q\frac{(g-\tau+m)^{g-\tau+m}}{m^m(g-\tau)^{g-\tau}} > a + 1 \text{ if } m \text{ is odd} \tag{2.14}$$

and

$$q\frac{(g-\sigma+m)^{g-\sigma+m}}{m^m(g-\sigma)^{g-\sigma}} > b \text{ if } m \text{ is even.} \tag{2.15}$$

Then (2.9) is oscillatory.

Proof. For $\lambda \neq 1$ we have

$$\frac{F_5(\lambda)\lambda^\sigma}{(\lambda-1)^m} = (\lambda^\sigma + a\lambda^{-\tau+\sigma} - b) + \frac{q\lambda^{-g+\sigma} + p\lambda^{h+\sigma}}{(\lambda-1)^m}. \tag{2.16}$$

Now, we consider the following three cases:

Case 1: m is even or odd and $\lambda > 1$.

From (2.16), it follows that

$$\frac{F_5(\lambda)\lambda^\sigma}{(\lambda-1)^m} \geq p\frac{\lambda^{h+\sigma}}{(\lambda-1)^m} + \lambda^\sigma + a\lambda^{\sigma-\tau} - b$$

As in the proof of Theorem 2.1–Case 1, we have

$$\frac{F_5(\lambda)\lambda^\sigma}{(\lambda-1)^m} \geq p\frac{(\frac{h+\sigma}{h+\sigma-m})^{h+\sigma}}{(\frac{m}{h+\sigma-m})^m} + 1 + a - b > 0.$$

Case 2: m is odd and $0 < \lambda < 1$.

In this case we have

$$\begin{aligned} -\frac{F_5(\lambda)\lambda^\tau}{(\lambda-1)^m} &= \frac{F_5(\lambda)\lambda^\tau}{(1-\lambda)^m} \\ &= \frac{q\lambda^{\tau-g} + p\lambda^{h+\tau}}{(1-\lambda)^m} - \lambda^\tau - a + b\lambda^{\tau-\sigma} \\ &\geq q\frac{\lambda^{-(g-\tau)}}{(1-\lambda)^m} - 1 - a \\ &\geq q\frac{(\frac{g-\tau}{g-\tau+m})^{-(g-\tau)}}{(\frac{m}{g-\tau+m})^m} - 1 - a > 0. \end{aligned}$$

Case 3: m is even and $0 < \lambda < 1$.

From (2.16), it follows that

$$\begin{aligned} \frac{F_5(\lambda)\lambda^\sigma}{(1-\lambda)^m} &\geq q\frac{\lambda^{-(g-\sigma)}}{(1-\lambda)^m} + \lambda^\sigma + a\lambda^{\sigma-\tau} - b \\ &\geq q\frac{(\frac{g-\sigma}{g-\sigma+m})^{-(g-\sigma)}}{(\frac{m}{g-\sigma+m})^m} - b > 0. \end{aligned}$$

The rest of the proof is similar to that of Theorem 2.1 and hence is omitted. □

Theorem 2.4. *Assume that* $a > b$, $a > 1$, $\sigma > \tau$ *and* $h - \tau > m$. *If*

$$p\frac{(h-\tau)^{h-\tau}}{m^m(h-\tau-m)^{h-\tau-m}} > a - b, \tag{2.17}$$

$$q\frac{(g+m)^{g+m}}{m^m g^g} > 1 + b \text{ when } m \text{ is odd} \tag{2.18}$$

and

$$q\frac{(g+\tau+m)^{g+\tau+m}}{m^m(g+\tau)^{g+\tau}} > a - 1 \text{ when } m \text{ is even,} \tag{2.19}$$

then (2.10) *is oscillatory.*

Proof. For $\lambda \neq 1$ we find

$$\frac{F_6(\lambda)\lambda^{-\tau}}{(\lambda-1)^m} = (\lambda^{-\tau} - a + b\lambda^{\sigma-\tau}) + \frac{q\lambda^{-(g+\tau)} + p\lambda^{h-\tau}}{(\lambda-1)^m}. \tag{2.20}$$

Now, we consider the following three cases:

Case 1: m is even or odd and $\lambda > 1$.

In this case we have

$$\begin{aligned}\frac{F_6(\lambda)\lambda^{-\tau}}{(\lambda-1)^m} &\geq p\frac{\lambda^{h-\tau}}{(\lambda-1)^m}+\lambda^{-\tau}-a+b\lambda^{\sigma-\tau}\\ &\geq p\frac{(\frac{h-\tau}{h-\tau-m})^{h-\tau}}{(\frac{m}{h-\tau-m})^m}-a+b>0.\end{aligned}$$

Case 2: m is odd and $0 < \lambda < 1$.

From (2.12), we have

$$\begin{aligned}-\frac{F_6(\lambda)}{(\lambda-1)^m} &= \frac{F_6(\lambda)}{(1-\lambda)^m}\\ &= \frac{q\lambda^{-g}+p\lambda^h}{(1-\lambda)^m}-1+a\lambda^\tau-b\lambda^\sigma\\ &\geq q\frac{\lambda^{-g}}{(1-\lambda)^m}-1-b\\ &\geq q\frac{(\frac{g}{g+m})^{-g}}{(\frac{m}{g+m})^m}-1-b>0.\end{aligned}$$

Case 3: m is even and $0 < \lambda < 1$.

It follows from (2.20) that

$$\begin{aligned}\frac{F_6(\lambda)\lambda^{-\tau}}{(\lambda-1)^m} &= \frac{F_6(\lambda)\lambda^{-\tau}}{(1-\lambda)^m}\\ &\geq q\frac{\lambda^{-(g+\tau)}}{(1-\lambda)^m}+\lambda^{-\tau}-a+b\lambda^{\sigma-\tau}\\ &\geq q\frac{(\frac{g+\tau}{g+\tau+m})^{-(g+\tau)}}{(\frac{m}{g+\tau+m})^m}+1-a>0.\end{aligned}$$

The rest of the proof is similar to that of Theorem 2.1 and hence is omitted. □

Next, we consider (1.3), where a, b and p are nonnegative real constants, q is a positive constant, g is a positive integer and h, τ and σ are nonnegative integers. In fact, the case when m is even is obvious. Therefore, we shall only consider (1.3) when m is odd and present the following result.

Theorem 2.5. *Suppose m is odd and $g > \tau$. If*

$$q\frac{(g-\tau+m)^{g-\tau+m}}{m^m(g-\tau)^{g-\tau}} > 1+a+b, \tag{2.21}$$

then (1.3) *is oscillatory.*

Proof. Since m is odd, it follows that $F_3(\lambda) > 0$ for $\lambda > 1$. Hence it remains to prove that $F_3(\lambda) > 0$ holds also for $0 < \lambda < 1$. Indeed

$$\begin{aligned}
-\frac{F_3(\lambda)\lambda^\tau}{(\lambda-1)^m} &= \frac{F_3(\lambda)\lambda^\tau}{(1-\lambda)^m} \\
&= \frac{q\lambda^{-(g-\tau)} + p\lambda^{h+\tau}}{(1-\lambda)^m} - \lambda^\tau - a - b\lambda^{\tau+\sigma} \\
&\geq \frac{q\lambda^{-(g-\tau)}}{(1-\lambda)^m} - (1+a+b) \\
&\geq q\frac{(\frac{g-\tau}{g-\tau+m})^{-(g-\tau)}}{(\frac{m}{g-\tau+m})^m} - (1+a+b) > 0.
\end{aligned}$$

Thus, $F_3(\lambda) > 0$ for all $\lambda \in \mathbb{R}^+$, i.e., (1.7) has no positive roots. So, the conclusion of the theorem follows from Lemma 2.1. □

Next, we consider the neutral difference equations which are of the same type of (1.3), namely

(2.22) $$\Delta^m \left(x(k) + ax[k-\tau] + bx[k-\sigma]\right) + qx[k-g] + px[k+h] = 0$$

and

(2.23) $$\Delta^m \left(x(k) + ax[k+\tau] + bx[k+\sigma]\right) + qx[k-g] + px[k+h] = 0,$$

where the coefficients a, b, p and q and the deviations τ, σ, g and h are as in (1.3). The characteristic equations of (2.22) and (2.23) are respectively

(2.24) $$F_7(\lambda) := (\lambda-1)^m \left[1 + a\lambda^{-\tau} + b\lambda^{-\sigma}\right] + q\lambda^{-g} + p\lambda^h = 0$$

and

(2.25) $$F_8(\lambda) := (\lambda-1)^m \left[1 + a\lambda^{\tau} + b\lambda^{\sigma}\right] + q\lambda^{-g} + p\lambda^h = 0.$$

Theorem 2.6. *Assume m is odd and $g > \tau \geq \sigma$. If condition (2.21) holds, then (2.22) is oscillatory.*

Proof. As in the proof of Theorem 2.5, we only need to show that $F_7(\lambda) > 0$ for $0 < \lambda < 1$. From (2.24) it follows that

$$\begin{aligned}
-\frac{F_7(\lambda)\lambda^\tau}{(\lambda-1)^m} &= \frac{F_7(\lambda)\lambda^\tau}{(1-\lambda)^m} \\
&= \frac{q\lambda^{-(g-\tau)} + p\lambda^{h+\tau}}{(1-\lambda)^m} - (\lambda^\tau + a + b\lambda^{\tau-\sigma}) \\
&\geq \frac{q\lambda^{-(g-\tau)}}{(1-\lambda)^m} - (1+a+b)
\end{aligned}$$

The rest of the proof is similar to that of Theorem 2.5 and hence is omitted. □

Theorem 2.7. *If m is odd and*

(2.26) $$q\frac{(g+m)^{g+m}}{m^m g^g} > 1 + a + b,$$

then (2.23) is oscillatory.

Proof. It suffices to prove that $F_8(\lambda) > 0$ for $0 < \lambda < 1$. From (2.25), it follows that

$$\begin{aligned} -\frac{F_8(\lambda)}{(\lambda-1)^m} &= \frac{F_8(\lambda)}{(1-\lambda)^m} \\ &= \frac{q\lambda^{-g} + p\lambda^h}{(1-\lambda)^m} - (1 + a\lambda^\tau + b\lambda^\sigma) \\ &\geq \frac{q\lambda^{-g}}{(1-\lambda)^m} - (1 + a + b). \end{aligned}$$

The rest of the proof is similar to that of Theorem 2.5 and hence is omitted. □

Next, we consider (1.4), where a, b are nonnegative real numbers and $a+b > 0$, p, q are positive real numbers, τ, σ are nonnegative integers and g and h are positive integers.
Now, we establish the following result.

Theorem 2.8. *Suppose that $g > \tau$ and $h > \sigma + m$. If*

$$p\frac{(h-\sigma)^{h-\sigma}}{m^m(h-\sigma-m)^{h-\sigma-m}} > a + b, \tag{2.27}$$

$$q\frac{(g+m)^{g+m}}{m^m g^g} > 1 - a \text{ when } m \text{ is odd} \tag{2.28}$$

and

$$q\frac{(g-\tau+m)^{g-\tau+m}}{m^m(g-\tau)^{g-\tau}} > a + b \text{ when } m \text{ is even,} \tag{2.29}$$

then (1.4) *is oscillatory.*

Proof. For $\lambda \neq 1$, we have

$$\frac{F_4(\lambda)\lambda^{-\sigma}}{(\lambda-1)^m} = \frac{q\lambda^{-(g+\sigma)} + p\lambda^{h-\sigma}}{(\lambda-1)^m} + \lambda^{-\sigma} - a\lambda^{-(\tau+\sigma)} - b. \tag{2.30}$$

Now, we consider the following three cases:

Case 1: m is even or odd and $\lambda > 1$.
From (2.30), it follows that

$$\begin{aligned} \frac{F_4(\lambda)\lambda^{-\sigma}}{(\lambda-1)^m} &\geq p\frac{\lambda^{h-\sigma}}{(\lambda-1)^m} + \lambda^{-\sigma} - a\lambda^{-(\tau+\sigma)} - b \\ &\geq p\frac{(\frac{h-\sigma}{h-\sigma-m})^{h-\sigma}}{(\frac{m}{h-\sigma-m})^m} - a - b > 0. \end{aligned}$$

Case 2: m is odd and $0 < \lambda < 1$.

In this case we have

$$\begin{aligned}
-\frac{F_4(\lambda)}{(\lambda-1)^m} &= \frac{F_4(\lambda)}{(1-\lambda)^m} \\
&= \frac{q\lambda^{-g}+p\lambda^h}{(1-\lambda)^m} - (1-a\lambda^{-\tau}-b\lambda^{\sigma}) \\
&\geq q\frac{\lambda^{-g}}{(1-\lambda)^m} - (1-a\lambda^{-\tau}-b\lambda^{\sigma}) \\
&\geq q\frac{(\frac{g}{g+m})^{-g}}{(\frac{m}{g+m})^m} - 1 + a > 0.
\end{aligned}$$

Case 3: m is even and $0<\lambda<1$.

In this case, we have

$$\begin{aligned}
\frac{F_4(\lambda)\lambda^{\tau}}{(\lambda-1)^m} &= \frac{F_4(\lambda)\lambda^{\tau}}{(1-\lambda)^m} \\
&= \frac{q\lambda^{-(g-\tau)}+p\lambda^{h+\tau}}{(1-\lambda)^m} + \lambda^{\tau} - a - b\lambda^{\tau+\sigma} \\
&\geq q\frac{\lambda^{-(g-\tau)}}{(1-\lambda)^m} + \lambda^{\tau} - a - b\lambda^{\tau+\sigma} \\
&\geq q\frac{(\frac{g-\tau}{g-\tau+m})^{-(g-\tau)}}{(\frac{m}{g-\tau+m})^m} - a - b > 0.
\end{aligned}$$

Cases 1–3 and $F_4(1)>0$ imply $F_4(\lambda)>0$ for $\lambda\in\mathbb{R}^+$, i.e., (1.8) has no positive roots. The results follows by applying Lemma 2.1. This completes the proof. □

Remark 2.1. Since the function $f(x)=ax^{-\tau}+bx^{\sigma}$, $0<x<1$ and $0<\frac{a\tau}{b\sigma}<1$ has a local minimum at $x=\left(\frac{a\tau}{b\sigma}\right)^{\frac{1}{\tau+\sigma}}$, we see that condition (2.28) can be replaced by the following weaker condition, namely

$$q\frac{(g+m)^{g+m}}{m^m g^g} > 1 - a\left(\frac{a\tau}{b\sigma}\right)^{-\frac{\tau}{\tau+\sigma}} - b\left(\frac{a\tau}{b\sigma}\right)^{\frac{\sigma}{\tau+\sigma}}. \tag{2.31}$$

The other conditions can be improved similarly. The details are left to the reader.

Finally, we consider equations of the same type of (1.4), namely,

$$\Delta^m\left(x(k)-ax[k-\tau]-bx[k-\sigma]\right)+qx[k-g]+px[k+h]=0 \tag{2.32}$$

and

$$\Delta^m\left(x(k)-ax[k+\tau]-bx[k+\sigma]\right)+qx[k-g]+px[k+h]=0, \tag{2.33}$$

where the coefficients a, b, p, q and the deviations g, h, τ and σ are as in (1.4). The characteristic equations of (2.32) and (2.33) are respectively,

$$F_9(\lambda) := (\lambda-1)^m\left[1-a\lambda^{-\tau}-b\lambda^{-\sigma}\right]+q\lambda^{-g}+p\lambda^h=0 \tag{2.34}$$

and

$$F_{10}(\lambda) := (\lambda-1)^m\left[1-a\lambda^{\tau}-b\lambda^{\sigma}\right]+q\lambda^{-g}+p\lambda^h=0. \tag{2.35}$$

Theorem 2.9. *Suppose that $g > \tau \geq \sigma$ and $h + \sigma > m$. If*

$$p\frac{(h+\sigma)^{h+\sigma}}{m^m(h+\sigma-m)^{h+\sigma-m}} > a+b-1, \tag{2.36}$$

$$q\frac{(g+m)^{g+m}}{m^m g^g} > 1-a-b \text{ when } m \text{ is odd} \tag{2.37}$$

and

$$q\frac{(g-\tau+m)^{g-\tau+m}}{m^m(g-\tau)^{g-\tau}} > a+b \text{ when } m \text{ is even,} \tag{2.38}$$

then (2.32) *is oscillatory.*

Proof. For $\lambda \neq 1$, we have

$$\frac{F_9(\lambda)\lambda^\sigma}{(\lambda-1)^m} = \frac{q\lambda^{-(g-\sigma)} + p\lambda^{h+\sigma}}{(\lambda-1)^m} + \lambda^\sigma - a\lambda^{-(\tau-\sigma)} - b. \tag{2.39}$$

Once again, we consider the following three cases:

Case 1: m is even or odd and $\lambda > 1$.

In this case, we obtain

$$\begin{aligned}\frac{F_9(\lambda)\lambda^\sigma}{(\lambda-1)^m} &\geq p\frac{\lambda^{h+\sigma}}{(\lambda-1)^m} + 1 - a - b\\ &\geq p\frac{(\frac{h+\sigma}{h+\sigma-m})^{h+\sigma}}{(\frac{m}{h+\sigma-m})^m} + 1 - a - b > 0.\end{aligned}$$

Case 2: m is odd and $0 < \lambda < 1$.

From (2.34), it follows that

$$\begin{aligned}-\frac{F_9(\lambda)}{(\lambda-1)^m} &= \frac{F_9(\lambda)}{(1-\lambda)^m}\\ &= \frac{q\lambda^{-g} + p\lambda^h}{(1-\lambda)^m} - 1 + a\lambda^{-\tau} + b\lambda^{-\sigma}\\ &\geq q\frac{\lambda^{-g}}{(1-\lambda)^m} - 1 + a + b\\ &\geq q\frac{(\frac{g}{g+m})^{-g}}{(\frac{m}{g+m})^m} - 1 + a + b > 0.\end{aligned}$$

Case 3: m is even and $0 < \lambda < 1$.

In this case, we find

$$\begin{aligned}\frac{F_9(\lambda)\lambda^\tau}{(\lambda-1)^m} &= \frac{q\lambda^{-(g-\tau)} + p\lambda^{h+\tau}}{(1-\lambda)^m} + \lambda^\tau - a - b\lambda^{\tau-\sigma}\\ &\geq q\frac{\lambda^{-(g-\tau)}}{(1-\lambda)^m} - a - b\\ &\geq q\frac{(\frac{g-\tau}{g-\tau+m})^{-(g-\tau)}}{(\frac{m}{g-\tau+m})^m} - a - b > 0.\end{aligned}$$

The rest of the proof is similar to that of Theorem 2.8 and hence is omitted. □

Theorem 2.10. *Suppose that $h > \sigma + m$, $\sigma \geq \tau$ and $g > \tau$. If*

$$p\frac{(h-\sigma)^{h-\sigma}}{m^m(h-\sigma-m)^{h-\sigma-m}} > a+b, \tag{2.40}$$

$$q\frac{(g+m)^{g+m}}{m^m g^g} > 1 \text{ when } m \text{ is odd} \tag{2.41}$$

and

$$q\frac{(g-\tau+m)^{g-\tau+m}}{m^m(g-\tau)^{g-\tau}} > a+b \text{ when } m \text{ is even}, \tag{2.42}$$

then (2.33) *is oscillatory.*

Proof. For $\lambda \neq 1$, we have

$$\frac{F_{10}(\lambda)\lambda^{-\sigma}}{(\lambda-1)^m} = \frac{q\lambda^{-g-\sigma} + p\lambda^{h-\sigma}}{(\lambda-1)^m} + \lambda^{-\sigma} - a\lambda^{\tau-\sigma} - b. \tag{2.43}$$

We consider the following three cases:

Case 1: m is even or odd and $\lambda > 1$.
It follows from (2.43), we find

$$\begin{aligned} \frac{F_{10}(\lambda)\lambda^{-\sigma}}{(\lambda-1)^m} &\geq p\frac{\lambda^{h-\sigma}}{(\lambda-1)^m} - a - b \\ &\geq p\frac{(\frac{h-\sigma}{h-\sigma-m})^{h-\sigma}}{(\frac{m}{h-\sigma-m})^m} - a - b > 0. \end{aligned}$$

Case 2: m is odd and $0 < \lambda < 1$.
From (2.35), it follows that

$$\begin{aligned} -\frac{F_{10}(\lambda)}{(\lambda-1)^m} &= \frac{F_{10}(\lambda)}{(1-\lambda)^m} \\ &= \frac{q\lambda^{-g} + p\lambda^h}{(1-\lambda)^m} - 1 + a\lambda^\tau + b\lambda^\sigma \\ &\geq q\frac{\lambda^{-g}}{(1-\lambda)^m} - 1 \\ &\geq q\frac{(\frac{g}{g+m})^{-g}}{(\frac{m}{g+m})^m} - 1 > 0. \end{aligned}$$

Case 3: m is even and $0 < \lambda < 1$.

In this case, we find

$$\begin{aligned}\frac{F_{10}(\lambda)\lambda^{\tau}}{(\lambda-1)^m} &= \frac{F_{10}(\lambda)\lambda^{\tau}}{(1-\lambda)^m}\\ &= \frac{q\lambda^{-(g-\tau)}+p\lambda^{h+\tau}}{(1-\lambda)^m}+\lambda^{\tau}-a\lambda^{2\tau}-b\lambda^{\tau+\sigma}\\ &\geq q\frac{\lambda^{-(g-\tau)}}{(1-\lambda)^m}-a-b\\ &\geq q\frac{(\frac{g-\tau}{g-\tau+m})^{-(g-\tau)}}{(\frac{m}{g-\tau+m})^m}-a-b>0.\end{aligned}$$

The rest of the proof is similar to that of Theorem 2.8 and hence is omitted. □

3. Some General Remarks

1. When $m=1$ is in any of our theorems presented above, we see that the equation

$$\Delta x(k)+qx[k-g]=0, \tag{3.1}$$

where q and g are positive real constants, is oscillatory if

$$q\frac{(g+1)^{g+1}}{g^g}>1. \tag{3.2}$$

We note that condition (3.2) is the well-known sufficient condition for the oscillation of (3.1), see [9].

2. From the proof of Theorem 2.1, condition (2.4) can be replaced by

$$q\frac{(g+\sigma+m)^{g+\sigma+m}}{m^m(g+\sigma)^{g+\sigma}}>b \text{ if } m \text{ is even.} \tag{3.3}$$

Now, if $a=b=0$, we see that the even order equations

$$\Delta^m x(k)+qx[k-g]=0,\ \ (q>0 \text{ and } g\geq 0)$$

$$\Delta^m x(k)+px[k+h]=0\ (p>0 \text{ and } h\geq 0)$$

and the mixed equation

$$\Delta^m x(k)+qx[k-g]+px[k+h]=0\ (p>0,\ q\geq 0 \text{ or } p\geq 0,\ q>0 \text{ and } g,h\geq 0)$$

are obviously oscillatory without imposing any extra conditions on p, q, g and h rather than those given above.

The conclusion can also be obtained from the rest of our results.

3. It is easy to see that Theorems 2.1–2.10 are either improved or else similar to our earlier corresponding results in [3, 2, 6].

4. By using the technique presented here, we can obtain more oscillation criteria for the equations considered and other with p and q are negative real numbers.

5. It is easy to construct examples showing that our criteria for oscillation of the equations considered are essentially wider than our oscillation criteria in [3, 2, 6].

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